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One-dimensional disordered magnetic Ising systems: A new approach

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We reconsider the problem of a one-dimensional Ising model with an arbitrary nearest-neighbor random exchange integral, temperature, and random magnetic field in each site. A convenient formalism is developed that reduces the partition function to a recurrence equation, which is convenient both for numerical as well as for analytical approaches. We have calculated asymptotic expressions for an ensemble averaged free energy and the averaged magnetization in the case of strong and weak couplings in external constant magnetic field. With a random magnetic field at each site in addition to nearest-neighbor random exchange integrals we also evaluated the free energy. We show that the zeros of the partition function for the Ising model in the complex external magnetic field plane formally coincide with the singularities of the real part of electron's transmission amplitude through the chain of δ-function potentials.

1 Introduction In the theory of disordered one-dimensional (1D) systems, many problems in thermodynamics and quantum transport can be reduced to the evolution of the product of transfer matrices [1–4]. The method is efficient in evaluation of many physical quantities (e.g., electron scattering matrix elements, energy spectrum, localization length, partition function, etc.) in finite periodic systems with complex, but identical unit cells. In such cases it is straightforward to diagonalize the product of transfer matrices. This method also allows us, in the presence of any kind of disorder (e.g., in the Anderson model with diagonal and/or off-diagonal disorder, in the case of inhomogeneous Ising alloys, or in quasiperiodic systems), to carry out numerical calculations very effectively. Unfortunately the method becomes almost useless for solving the problem analytically in the general case when the number of impurities becomes very large. This is connected with the fact that the product of the individual transfer matrices do not commute and one needs to calculate the product of all transfer matrices which exponentially increases with their number. This is the main difficulty of the transfer matrices method and arose in many studies in the theory of 1D disordered systems and particularly of calculation of partition function in 1D disordered magnetic Ising systems in the intermediate region, i.e., when the magnetic field (or temperature) is not very high or low (see, e.g., Refs. [5, 6]). The situation becomes more complicated when we deal with two or more random parameters as in the case of 1D Ising inhomogeneous model with random magnetic field at each site in addition to nearest-neighbor random exchange integrals. A new approach which avoids direct calculation of the product of arbitrary 2×2 matrices in many 1D problems was presented in Ref. [3]. The author has shown that the product of matrices can be reduced to a recurrence relation directly for the exponent and thus maps many 1D problems on each other. Nevertheless the method developed in Ref. [3] is more convenient, in our view, for large systems, where the boundary and initial conditions are not as important.

Another useful method, based on the exact calculation of the Green's function of a quantum particle in a given potential, to investigate the behavior of electrons in one and multi-channel random chains was developed in Refs. [7, 8]. This so-called characteristic determinant method, in our view, has some advantages compared with the methods used in Refs. [1–3]. First of all, this method makes it possible to reduce the calculation of the product of finite and arbitrary matrices to the calculation of the determinant D_N (or inverse transmission amplitude) of the order $N \times N$ (N is the number of individual scattering sites or atoms in the chain). Secondly, knowing the explicit form of the Green's function of a quantum particle in a given potential makes it possible to calculate the average density of states over the sample as well as the characteristic barrier tunneling time [9] and represent them by the determinant D_N.

The purpose of the present paper is to extend the method of characteristic determinant to the general case of the disordered 1D Ising model with an arbitrary nearest-neighbor random exchange integral, temperature, and random magnetic field in each site. Magnetic realization of the latter is typically dilute antiferromagnetism in uniform field. We explore further the analogy between the phase coherent charge transport through single channel systems and 1D Ising models; even though these two major areas are seemingly disparate and have evolved independently. Our main focus is to present a unified approach which will allow us first to recover all well-known results for 1D Ising model and second to derive simple analytical expressions for the ensemble averaged free energy $\langle F\rangle$ and for the averaged magnetization $\langle M\rangle$ in the intermediate region when the magnetic field (or temperature) is not very high or low. To the best of our knowledge there are no analytical expressions for $\langle F\rangle$ and $\langle M\rangle$ in this regime because, as we mentioned above, all available methods are more convenient for numerical simulations only.

2 Model and derivation of the main relations

We consider a 1D chain of N spins with an arbitrary interaction between the nearest atoms in the presence of a random magnetic field (inhomogeneous model). The model is defined by the Hamiltonian

$$H = -\sum_{j=1}^{N} h_j\sigma_j - \sum_{j=1}^{N} I_j\sigma_j\sigma_{j+1}, \tag{1}$$

where σ_j takes the values ± 1, I_j is the nearest-neighbor random exchange integral, and h_j is the random magnetic field at site j. We assume that all the h_j's and I_j's are independent random variables with the distribution functions $f(h)$ and $g(I)$, respectively. The aim, as always, is to calculate the partition function in the thermodynamic limit $N \to \infty$.

The partition function of the system is given by

$$Z^{(N)} = \sum_{\sigma_1=\pm1} \cdots \sum_{\sigma_N=\pm1} \exp\left\{\beta\sum_{j=1}^{N} h_j\sigma_j + \beta\sum_{j=1}^{N-1} I_j\sigma_j\sigma_{j+1}\right\}, \tag{2}$$

where $\beta = 1/kT$ is the inverse temperature.

It is well known that in the homogeneous case, when all the nearest-neighbor exchange integrals I_j are equal, the partition function can be obtained by means of the transfer matrices technique [10, 11]. Following this method, one can show that in general case, as discussed here, the partition function $Z^{(N)}$, Eq. (2), corresponding to a closed sequence of N spins ($\sigma_1 = \sigma_{N+1}$) can be written in the following form:

$$\begin{aligned} Z^{(N)} &= \sum_{\sigma_1=\pm1} \cdots \sum_{\sigma_N=\pm1} P_1(\sigma_1,\sigma_2)P_2(\sigma_2,\sigma_3)\cdots P_N(\sigma_N,\sigma_1) \\ &= \mathrm{Tr}\, P^{(N)}, \end{aligned} \tag{3}$$

where

$$P^{(N)} = P_1P_2\cdots P_N = \begin{pmatrix} P_{11}^{(N)} & P_{12}^{(N)} \\ P_{21}^{(N)} & P_{22}^{(N)} \end{pmatrix}. \tag{4}$$

For the P_j we can use the following representation [3]:

$$P_j = [2\sinh(2\widetilde{I}_j)]^{1/2} \begin{pmatrix} \dfrac{e^{y_j}}{\cos x_j} & e^{y_j}\tan x_j \\ e^{-y_j}\tan x_j & \dfrac{e^{-y_j}}{\cos x_j} \end{pmatrix}, \tag{5}$$

where $y_j = \beta h_j$, $x_j = \arcsin(e^{-2\widetilde{I}_j})$, and $\widetilde{I}_j = \beta I_j$.

Now let us define unimodular matrices M_j

$$M_j = \begin{pmatrix} \dfrac{e^{y_j}}{\cos x_j} & e^{y_j}\tan x_j \\ e^{-y_j}\tan x_j & \dfrac{e^{-y_j}}{\cos x_j} \end{pmatrix}. \tag{6}$$

Then a successive matrix $M^{(N)}$ is an ordered product of the individual matrices M_j

$$M^{(N)} = M_1M_2\cdots M_N = \begin{pmatrix} M_{11}^{(N)} & M_{12}^{(N)} \\ M_{21}^{(N)} & M_{22}^{(N)} \end{pmatrix}, \tag{7}$$

with the matrix elements $M_{11}^{(N)}$ and $M_{22}^{(N)}$ given below

$$M_{11}^{(N)} = D_N\, e^{\sum_j^N y_j}, \tag{8}$$

$$M_{12}^{(N)} = \frac{D_N - \cos x_N D_{N-1}}{\sin x_N}\, e^{\sum_j^N y_j}. \tag{9}$$

Here, D_N is the characteristic determinant introduced in Ref. [7]. The matrix elements of $N \times N$ determinant D_N are determined by

$$\begin{aligned} (D_N)_{nl} = \det\Bigg| &\frac{\delta_{nl}}{\cos x_n} + [\theta(l-n) - \theta(n-l)](1-\delta_{nl}) \\ &\times e^{-\sum_{j=2}^{\max(l,n)} y_j} \tan x_l \Bigg|, \end{aligned} \tag{10}$$

where $\theta(z) = 1$ if $z > 0$ and 0 otherwise.

Note that the matrix elements $M_{22}^{(N)}$ and $M_{21}^{(N)}$ are obtained from Eqs. (8) and (9) by replacing $y_j \to -y_j$

$$M_{22}^{(N)} = M_{11}^{(N)}(-y_j), \tag{11}$$

$$M_{21}^{(N)} = M_{12}^{(N)}(-y_j). \tag{12}$$

Then it follows directly from the obtained expression (10) that the determinant D_N can be presented in tridiagonal Toeplitz form, where the only non-zero elements are the diagonal elements nn, and the nearest-neighbor elements $n \pm 1$

$$\det D_N = \begin{vmatrix} A_1 & b_1 & 0 & \cdots & \cdots & 0 \\ c_1 & A_2 & b_2 & 0 & \cdots & 0 \\ 0 & c_2 & A_3 & b_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{N-2} & A_{N-1} & b_{N-1} \\ 0 & \cdots & \cdots & 0 & c_{N-1} & A_N \end{vmatrix}. \tag{13}$$

c_{N-1}, b_{N-1}, and A_N are given by

$$c_{N-1} = -\frac{e^{-y_N}}{\cos x_{N-1}}(1 + \sin x_{N-1}); \quad c_0 = 0,$$

$$b_{N-1} = -\frac{e^{-y_N}\tan x_N}{\sin x_{N-1}}(1 - \sin x_{N-1}); \quad b_0 = 0, \tag{14}$$

$$A_N = \frac{1}{\cos x_N} + \frac{B_N}{\cos x_{N-1}}; \quad A_1 = \frac{1}{\cos x_1}$$

with

$$B_N \equiv c_{N-1}b_{N-1} = \frac{\tan x_N}{\tan x_{N-1}}\, e^{-2y_N}; \quad B_1 = 0.$$

By expanding in minors on the last row, we find for the determinant D_N the recurrence relation

$$D_N = A_N D_{N-1} - B_N D_{N-2}, \tag{15}$$

where the initial conditions are: $D_1 = 1/\cos x_1$, $D_0 = 1$, and $D_{-1} = 0$.

$D_{N-1(N-2)}$ is the determinant of the matrix of Eq. (10) in which the Nth or $(N-1)$th row and column are absent. Note that the analogous recurrence relation to Eq. (15), with different A_N and B_N, takes place in the calculation of the transmission amplitude of the electron through a random 1D system as in the case of an open system [7], as well as with the periodic conditions on the system, equivalent to joining the two ends of the line so as to form a circle [12].

In view of Eqs. (4)–(12), $Z^{(N)}$ can be written in terms of the $D_N(y_1, \ldots, y_N)$ and $D_N(-y_1, \ldots, -y_N)$

$$Z^{(N)} = \left[D_N(y_1, \ldots, y_N)\, e^{\sum_j^N y_j} + D_N(-y_1, \ldots, -y_N) \times e^{-\sum_j^N y_j} \right] \prod_{j=1}^{N} \left[2\sinh(2\widetilde{I}_j) \right]^{1/2}. \tag{16}$$

This formula represents the central result of our work. Together with Eq. (15), it expresses the partition function fully in terms of the individual atoms in the chain and allows one to compute, for an arbitrary y_j and $\widetilde{I}_j$, the partition function $Z^{(N)}$ and any other physical quantities, e.g., free energy $\langle F \rangle$, equilibrium magnetization $\langle M \rangle$, etc. It is worth noticing that using the transfer matrix techniques one can derive a recurrence relation directly for the ratios $Z^{(N)}(+)/Z^{(N)}(-)$ (for more details see Refs. [13, 14]), where $Z^{(N)}(+)$ and $Z^{(N)}(-)$ are the partition function of N spins with the last spin $s_n = +1$ and $s_n = -1$, respectively. This functional recursion relation, yielding the fixed distribution of the relation from which the free energy and other thermodynamic properties can be calculated, looks too complicated for the exact results.

In the following section, we analyze the limits of (16) and will provide an asymptotic expressions for $\langle F \rangle$ and for $\langle M \rangle$ in the case of strong and weak couplings, without any restriction on the external magnetic field h. We also analyzes the case of binary $\pm h_0$ magnetic field distribution.

3 Method of solution Before we proceed further and derive the partition function for an arbitrary external magnetic field, we briefly discuss a practical algorithm for solving the recurrence Eq. (15) for two well-known cases [10, 11].

3.1 Homogeneous Ising model In the homogeneous Ising model all nearest-neighbor random exchange integrals $\widetilde{I}_j$ and the random fields y_j are equal ($\widetilde{I}_1 = \cdots = \widetilde{I}_N \equiv \widetilde{I}$ and $y_1 = \cdots = y_N \equiv y$). This means that the quantities A_N and B_N (see Eq. 14) do not depend on indices N and that D_N can be presented in the form $D_N = c_1 U_1^N + c_2 U_2^N$, where U_1 and U_2 are the solutions of the equation

$$U^2 - AU + B = 0, \tag{17}$$

and c_1 and c_2 are assumed to satisfy the initial conditions

$$D_0 = c_1 + c_2 = 1$$

$$D_1 = c_1 U_1 + c_2 U_2 = \frac{1}{\cos x_1}.$$

Solving Eq. (17) and taking into account the initial conditions for c_1 and c_2 we obtain for D_N the following expression:

$$D_N = \frac{e^{-Ny}}{2\sqrt{e^{-4\widetilde{I}} + \sinh^2 y}\,\cos^N x} \times \left\{ e^{y}(\lambda_+^N - \lambda_-^N) - (\lambda_+^{N-1} - \lambda_-^{N-1})\cos^2 x \right\}, \tag{18}$$

where

$$\lambda_\pm = e^{\widetilde{I}}\left(\cosh y \pm \sqrt{e^{-4\widetilde{I}} + \sinh^2 y} \right). \tag{19}$$

If we formally define

$$\cosh\beta_1=\frac{\cosh y}{\cos x};\quad \cos x=\sqrt{1-\mathrm{e}^{-4\widetilde{I}}}>0,$$

then Eq. (18), after simple algebra, can be rewritten in the more compact form

$$D_N=\mathrm{e}^{-Ny}\left\{\cosh(N\beta_1)+\frac{\sinh y}{\cos x}\frac{\sinh(N\beta_1)}{\sinh\beta_1}\right\}. \qquad (20)$$

It is not difficult to see that the second term in the bracket is an odd function with respect to y and thus the partition function $Z^{(N)}$ (16) reads

$$Z^{(N)}=\left[D_N(y)\,\mathrm{e}^{Ny}+D_N(-y)\,\mathrm{e}^{-Ny}\right]\left(\frac{\mathrm{e}^{\widetilde{I}}}{\cos x}\right)^N$$
$$=\lambda_+^N+\lambda_-^N.$$

In the thermodynamic limit, when $N\to\infty$, $(\lambda_-/\lambda_+)^N\to 0$ we have

$$Z^{(N)}=\lambda_+^N, \qquad (21)$$

that is, the result which was obtained by Kramers and Wannier using the transfer matrices technique [10].

The partition function for the Ising model on a 1D translationally invariant lattice with complex unit cells with different exchange integrals $\widetilde{I}_j$ can be found in a similar way by solving the same recurrence relation (15) for D_N. Together with the periodicity of the system it allows us to find an analytical expression for the characteristic determinant D_{Mn} ($M\times n$ is equal to N, the total number of spins) and hence the partition function. In Section 4 we show that using the similarity between quantum electron transport in 1D chain and the Ising model, one can easily get the partition function in such a case.

3.2 Zero magnetic field The second case when the recurrence Eq. (15) can be calculated straightforwardly for an arbitrary $\widetilde{I}_j$ is zero magnetic field, i.e., $h_j=y_j=0$. In this case it is easy to verify that the solution of Eq. (15) has the form

$$D_N=\frac{1}{2}\left(\prod_{j=1}^{N}(a_j+b_j)+\prod_{j=1}^{N}(a_j-b_j)\right), \qquad (22)$$

where $a_j=1/\cos x_j$ and $b_j=\tan x_j$.

Thus, the partition function reads

$$Z^{(N)}=2^N\left(\prod_{j=1}^{N}\cosh\widetilde{I}_j+\prod_{j=1}^{N}\sinh\widetilde{I}_j\right), \qquad (23)$$

and as $N\to\infty$ this lead us to the other well-known result for the inhomogeneous Ising model at zero magnetic field [10]

$$Z^{(N)}=2^N\prod_{j=1}^{N}\cosh\widetilde{I}_j. \qquad (24)$$

3.3 An arbitrary constant external magnetic field Equations (21) and (24) are well known from the theory of the 1D Ising model [11] and are included only for completeness. Now we can go a step further and discuss the thermodynamic properties of the system, assuming that the nearest-neighbor exchange integrals $\widetilde{I}_j$ are independent random variables with the same probability distribution function $g(\widetilde{I})$. As for the external magnetic field we will assume that it is an arbitrary, but constant, i.e., $h_1=\cdots=h_N=h$. Although it is well known that the equilibrium properties of the system in an external magnetic field h cannot be evaluated exactly, i.e., it is not possible to find the analytical solution for $Z^{(N)}$ in the general case, we will show that in two limiting cases one can find new asymptotic expressions for $Z^{(N)}$.

First let us discuss the case of strong coupling which assumes that $\widetilde{I}_j\gg 1$ ($I_j\gg kT$). In this case the recurrence relation (15) can be solved in the following way. The initial conditions (see Eq. 14) for the recurrence relation (15) can be approximately replaced by $\cos x\approx 1$, $A_N\approx 1+B_N$, and $D_1\approx 1$. Then it is readily verified that the approximate solution of the recurrence relation (15), with above assumptions, is $D_N=1$. Substituting $D_N(y)=D_N(-y)=1$ into Eq. (16), we find for the partition function $Z_\mathrm{s}^{(N)}$

$$Z_\mathrm{s}^{(N)}\approx\left(\mathrm{e}^{Ny}+\mathrm{e}^{-Ny}\right)\prod_{j=1}^{N}\left[2\sinh 2\widetilde{I}_j\right]^{1/2}$$
$$\approx 2\,\mathrm{e}^{\sum_{j=1}^{N}\widetilde{I}_j}\cosh(Ny). \qquad (25)$$

We added a subscript s to indicate that we are discussing the strong coupling case. Note that the above equation reduces to Eq. (24) when we take magnetic field $y=0$ and $\widetilde{I}_j\gg 1$, as it should be.

Now, it is straightforward, for any external magnetic field h using Eq. (25) to evaluate the average free energy $\langle F_\mathrm{s}^{(N)}\rangle$ and the average magnetization $\langle M_\mathrm{s}^{(N)}\rangle$ for a given distribution function $f(\widetilde{I}_j)$. For simplicity we discuss the case when the nearest-neighbor random exchange integral $\widetilde{I}_j$ is distributed uniformly in an interval [0,W], i.e., distribution function $g(\widetilde{I}_j)=1$. After averaging we will get for $\langle F_\mathrm{s}^{(N)}\rangle$ and $\langle M_\mathrm{s}^{(N)}\rangle$, respectively

$$\langle F_s^{(N)} \rangle = -\frac{kT}{NW^N}\int_0^W \cdots \int_0^W \ln Z_s^{(N)} \mathrm{d}I_1 \ldots \mathrm{d}I_N$$
$$\approx -\frac{W}{2} - \frac{kT}{N}\ln\cosh Ny, \quad (26)$$

$$\langle M_s^{(N)} \rangle = -\frac{1}{N}\left\langle \frac{\partial}{\partial y}\ln Z_s^{(N)} \right\rangle \approx -\tanh Ny. \quad (27)$$

In the low- and high-h limits the free energy (26) can be shown to be

$$\langle F_s^{(N)} \rangle \approx -\frac{W}{2} - \frac{N}{2kT}h^2 + \mathcal{O}(h^4) \quad (28)$$

and

$$\langle F_s^{(N)} \rangle \approx -h - \frac{kT}{N}\,\mathrm{e}^{-2Nh/kT} + \cdots, \quad (29)$$

respectively.

Note that similar asymptotic expressions for an average free energy $\langle F \rangle$ in the low- and high-h limits, using the different probability law for I_j, were also obtained in Ref. [5]. Nevertheless in Ref. [5] the mathematical method of calculation of the free energy, based on the evaluation of N transfer matrices did not allow their authors to get an analytical expression for an average free energy $\langle F \rangle$ in the intermediate region in contrast to our result (26) which is valid for arbitrary magnetic field h. For this reason both expansions in Ref. [5], like (28) and (29), were shown to be connected only numerically, through direct numerical evaluation of a product of N transfer matrices.

To complete all the methods of calculating the recurrence relation (Eq. (15)), we finally consider the case of weak coupling, i.e., when the average strength of the $\widetilde{I}_j \ll 1$ ($I_j \ll kT$). In this case we have $\sin x \approx 1$, $\cos x \approx 0$, $A_N \gg B_N$, and the recurrence relation can be written in the form

$$D_N = A_N D_{N-1}.$$

It is not difficult to see that the solution of the above recurrence relation has the form

$$D_N = A_1 \prod_{j=2}^{N} A_j$$

or

$$D_N = \frac{1}{\cos x_1}\prod_{j=2}^{N}\left(\frac{1}{\cos x_j} + \mathrm{e}^{-2y}\tan x_j\right).$$

Substituting the above expression for D_N in Eq. (16) gives

$$Z_w^{(N)} = \mathrm{e}^{\widetilde{I}_1+y}\prod_{j=2}^{N} 2\cosh(\widetilde{I}_j + y) + \mathrm{e}^{\widetilde{I}_1-y}\prod_{j=2}^{N} 2\cosh(\widetilde{I}_j - y),$$

and because $\widetilde{I}_j \ll 1$ the final results for the partition function, free energy, and magnetization read, respectively

$$Z_w^{(N)} = (2\cosh y)^N \mathrm{e}^{\widetilde{I}_1}\prod_{j=2}^{N}\cosh\widetilde{I}_j$$
$$\approx (2\cosh y)^N\left(1 + \frac{1}{2}\sum_{j=1}^{N}\widetilde{I}_j^{\,2}\right), \quad (30)$$

$$\langle F_w^{(N)} \rangle \approx -kT\ln 2\cosh\frac{h}{kT} - \frac{W^2}{6kT}, \quad (31)$$

and

$$\langle M_w^{(N)} \rangle \approx -\tanh\frac{h}{kT}. \quad (32)$$

Note that the result, Eq. (32), is similar to the one which can be obtained from simple Ising model magnetization (per site)

$$M_{N\to\infty}^{(1)} = -\frac{\mathrm{e}^{2\widetilde{I}}\sinh\frac{h}{kT}}{\sqrt{1 + \mathrm{e}^{4\widetilde{I}}\sinh^2\frac{h}{kT}}}, \quad (33)$$

if one takes the limit of high temperature ($\widetilde{I} \to 0$). This means that at high temperature the disorder does not play an essential role.

In Fig. 1, we plot average free energy for strong (dashed line) and weak (dot line) coupling regimes, defined by Eqs. (26) and (31), as a function of the disorder W (strength of random exchange integral). It can be seen that the expected behavior for these limits is obtained and that the curves are fitted well with the numerical result, obtained from Eq. (16)

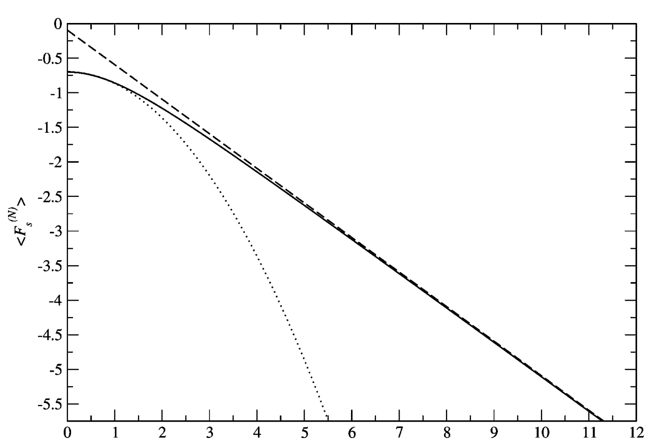

Figure 1 Average free energy as a function of disorder W (strength of random exchange integral). Solid, dashed, and dot curves are given by Eqs. (16), (28), and (31), respectively. $N = 100$ and $h = 0.1$.

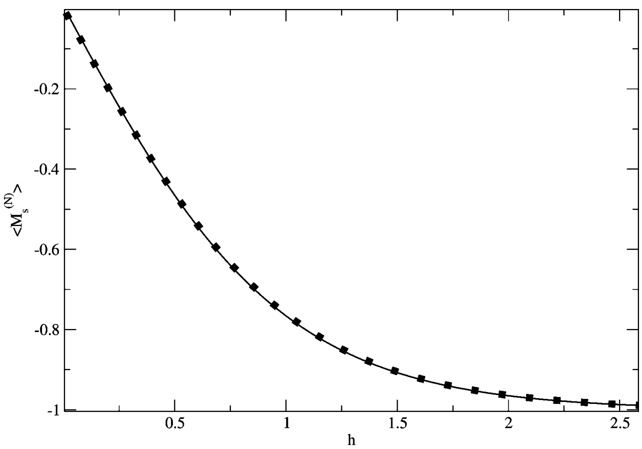

Figure 2 Average magnetization (per site) as a function of disorder magnetic field h (dotted line, Eq. (32)). Solid curve is obtained from Eq. (16). $N = 100$ and $w = 0.02$.

(solid line). The value of external magnetic field is $h = 0.1$ and number of the spins in the chain is $N = 100$.

In Fig. 2, we have plotted the average magnetization (per site) for the weak coupling regime (dotted curve), defined by Eq. (32), as a function of the external magnetic h. The solid line is the numerical result based on Eq. (16). Figure 2 shows that the analytical expression (32) agrees very well with the exact expression (solid line) in the whole range $0 < h < 2.5$ for the values of disorder $w = 0.02$ and number of spins in the chain $N = 100$.

Let us finally present the average magnetization (per site) for strong coupling regime as a function of external field h (see Fig. 3). The dotted line is the limiting expression, Eq. (27), which asymptotically follows to the numerical result, based on Eq. (16). One can see that for $h \geq 0.8$ both curves practically coincide. The difference between the two curves for small h in Fig. 3 connected with the fact that in expression (25) we ignored the terms which are proportional to $e^{-4\widetilde{I}_j}$.

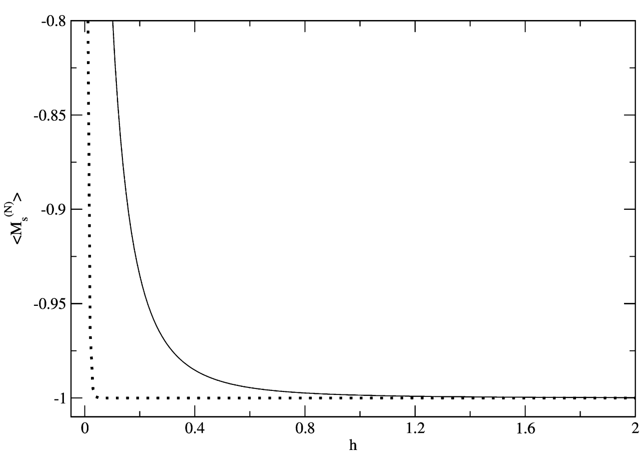

Figure 3 Same as Fig. 2, but for strong coupling. Dotted line is the average magnetization and is given by Eq. (32). Solid curve based on numerical calculation (Eq. (16)).

3.4 Random magnetic field Our objective in this subsection is to extend our previous calculations of partition functions $Z_{\mathrm{s}}^{(N)}$ and $Z_{\mathrm{w}}^{(N)}$ (see Eqs. (25) and (30)) when we have random magnetic field at each site in addition to nearest-neighbor random exchange integrals. We proceed along the same line as in Subsection 3.3, and show that only minor modifications of the final expressions, Eqs. (25) and (30), are required in order to include a random magnetic field y_j. Indeed, for $Z_{\mathrm{s,RMF}}^{(N)}$ we will get a similar expression to Eq. (25)

$$Z_{\mathrm{s,RMF}}^{(N)} \approx \left(\mathrm{e}^{\sum_{j=1}^{N} \widetilde{I}_j} + \mathrm{e}^{-\sum_{j=1}^{N} \widetilde{I}_j} \right) \prod_{j=1}^{N} \left[2 \sinh 2\widetilde{I}_j \right]^{1/2}$$
$$\approx 2\, \mathrm{e}^{\sum_{j=1}^{N} \widetilde{I}_j} \cosh \sum_{j=1}^{N} y_j. \tag{34}$$

Here, subscript RMF stands for "random magnetic field". From the comparison of the above partition function in the presence of random magnetic field y_j with the expression of partition function when magnetic field is constant (but arbitrary, see Eq. (25)) one can see that in Eq. (34) Ny is replaced by the sum over y_j. The reason for this replacement in such a simple way is connected with the fact that in the limit of strong coupling both parameters, $\widetilde{I}_j$ and y_j, are not correlated, statistically independent of each other and can be easily decoupled. After ensemble averaging over the random exchange integral $\widetilde{I}_j$, distributed uniformly in an interval $[0,W]$, the average free energy for this Ising model takes the form

$$\left\langle F_{\mathrm{s,RMF}}^{(N)} \right\rangle = -\frac{kT}{NW^N} \int_0^W \cdots \int_0^W \ln Z_{\mathrm{s,RMF}}^{(N)} \, \mathrm{d}I_1 \ldots \mathrm{d}I_N$$
$$\approx -\frac{W}{2} - \frac{kT}{N} \ln \cosh \sum_{j=1}^{N} y_j. \tag{35}$$

In the next stage, we need to average $\langle F_{\mathrm{s,RMF}}^{(N)} \rangle$ over the random field y_j and so for given distribution $f(y_j)$ calculate integral over variables y_j. Here, we want to emphasize the following point. The presence of the summation over y_j in the logarithmical function in $\langle F_{\mathrm{s,RMF}}^{(N)} \rangle$ does not allow us to get a closed analytical expression and one has to rely on numerical solution of Eq. (35). Thus, the analytical calculations no longer possible without further simplifying assumptions, connected with the magnitude of the random field y_j. To set an analytical expression for an average free energy over the random y_j let us discuss, as we did in Subsection 3.3, the two limiting cases of random magnetic field y_j, i.e., $y_j \gg 1$ and $y_j \ll 1$. For simplicity, we will focus on so-called binary distribution function which describes the discrete random fields where $h_j = +h_0$ and $h_j = -h_0$ each occur with probability 1/2 (see, e.g., Refs. [13, 14]). In this case

$$f(h) = \frac{1}{2} \left[\delta(h - h_0) + \delta(h + h_0) \right]. \tag{36}$$

Assuming that all $y_j \gg 1$ and after Taylor expansion of the logarithmic function in Eq. (35) and using the above binary distribution function, one can get the following expression for free energy

$$\overline{\langle F^{(N)}_{\mathrm{s,RMF}} \rangle} \approx -\frac{kT}{N} \cosh^N \frac{2h_0}{kT}. \tag{37}$$

Here, the over bar denotes averaging over the ensemble of chains with the distribution function $f(h)$ (Eq. 36).

In the low magnetic field limit, i.e., $y_j \ll 1$, we arrive at

$$\overline{\langle F^{(N)}_{\mathrm{s,RMF}} \rangle} \approx -\frac{W}{2} - \frac{h_0^2}{2kT} + \mathcal{O}(h^4). \tag{38}$$

The case of the weak coupling can be treated in the same way as we did in Subsection 3.3 (see Eq. 31). The final result for $\overline{\langle F^{(N)}_{\mathrm{w,RMF}} \rangle}$ is

$$\overline{\langle F^{(N)}_{\mathrm{w,RMF}} \rangle} \approx -kT \ln 2 \cosh \frac{h_0}{kT} - \frac{W^2}{6kT}. \tag{39}$$

4 Zeros of the partition function As mentioned in Section 1 there is considerable similarity between the phase coherent charge transport through single channel system and 1D Ising model, in spite of the fact that these two major areas are seemingly disparate and have evolved independently. The resemblance runs even deeper when one considers the zeros of the partition function, Eq. (16), and analyzes the chaotic trajectories in the complex external magnetic field h plane (for more details, see Refs. [15, 16] and references therein).

To demonstrate the technique of complex y in calculation of the partition function, Eq. (16), in the case of a periodic system with many atoms in a unit cells we follow closely to Ref. [9] and write down the explicit form of Green's function poles (or characteristic determinant which is inversely proportional to the transmission amplitude through a general structure) in the case of generalized Kronig–Penny model

$$\tilde{D}_{n\times K} = \mathrm{e}^{iKkd} \left\{ \cos(K\beta d) + i\, \mathrm{Im}\left\{\mathrm{e}^{-ikd}\tilde{D}_n\right\} \frac{\sin(K\beta d)}{\sin(\beta d)} \right\}, \tag{40}$$

where $\tilde{D}_n$ is the characteristic determinant for one unit cell, d is the lattice period, K is the number of cells, and $n \times K$ is equal to N, the total number of δ-potentials. β plays the role of quasimomentum for the GKP and is given by the equation

$$\cos(\beta d) = \mathrm{Re}\left\{\mathrm{e}^{-ikd}\tilde{D}_n\right\}. \tag{41}$$

When the modulus of the RHS of Eq. (41) turns out to be greater than 1, β has to be taken as imaginary. This situation corresponds to a forbidden electron's energy gap in an infinite system.

Now the first step is to replace in Eq. (40) kd by iy^* and βd by β_n. Then we will get

$$D_{n\times K} = \mathrm{e}^{-Ny} \left\{ \cosh(K\beta_n) + iL \frac{\sinh(K\beta_n)}{\sinh(\beta_n)} \right\}, \tag{42}$$

where

$$\cosh \beta_n = \mathrm{Re}\left\{\mathrm{e}^{-iny^*} D_n(y \to -iy^*)\right\}\big|_{y*\to iy} \tag{43}$$

and

$$L = \mathrm{Im}\left\{\mathrm{e}^{-iny^*} D_n(y \to -iy^*)\right\}\big|_{y^*\to iy}. \tag{44}$$

The second key step, after calculating the $\cosh \beta_n$ and the imaginary part of L, makes another analytical continuation by replacing $y^* \to iy$.

It can be checked directly that the second term in Eq. (42) is always an odd function with respect to y and thus the final expression for the partition function can be presented

$$Z^{(n\times K)} = \left\{ 2\cosh(K\beta_n) \prod_{j=1}^{n} [2\sinh 2\tilde{I}_j]^{K/2} \right\}$$
$$= (\lambda_+^{(n)})^K + (\lambda_-^{n})^K, \tag{45}$$

with

$$\lambda_\pm^{(n)} = \mathrm{e}^{\pm\beta_n} \prod_{j=1}^{n} \mathrm{e}^{\tilde{I}_j} \cos x_j. \tag{46}$$

The validity of the expression $Z^{(n\times K)}$ for the case of a periodic spin system with complex cells may be explicitly checked for small n, i.e., $n = 1, 2, \ldots$. For example, for $n = 1$, one has from Eqs. (15) and (43)

$$\cosh \beta_1 = \mathrm{Re}\left\{ \mathrm{e}^{-iy^*} \frac{1}{\cos x} \right\}\bigg|_{y^*\to iy} = \frac{\cosh y}{\cos x},$$

which yields Eq. (20).

For $n = 2$ one has

$$\cosh \beta_2 = \mathrm{Re}\left\{ \mathrm{e}^{-2iy^*}\left[\frac{1}{\cos x_1 \cos x_2} + \mathrm{e}^{2iy^*} \tan x_1 \tan x_2 \right]\right\}\bigg|_{y^*\to iy}$$
$$= \frac{\cosh 2y}{\cos x_1 \cos x_2} + \tan x_1 \tan x_2. \tag{47}$$

By inserting Eq. (47) into Eq. (45) we obtain partition function $Z^{(2\times K)}$. For illustrative purpose one can check that for magnetization per cell, in the thermodynamic limit, we have

$$M^{(2)}_{K\to\infty} = -\frac{1}{K}\frac{\partial}{\partial y} \ln Z^{(2\times K)}$$
$$= -\frac{\sinh 2y}{\sqrt{(\cos^2 y + \sin x_1 \sin x_2)^2 - \cos^2 x_1 \cos^2 x_2}}. \tag{48}$$

For identical nearest-neighbor exchange integrals, $x_1 = x_2$ the homogeneous case (Eq. 33) is recovered.

Concluding, note that the zeros of the partition function $Z^{(n\times K)} = (\lambda_+^{(n)})^K + (\lambda_-^n)^K = 0$ in the complex y plane can be found by replacing $\lambda_+^{(n)} = e^{iq\pi/K}\lambda_-^n$ $(-K<q \leq K$ is odd). The case of $n = 1$ (homogenous Ising model) was discussed in Ref. [15].

5 Conclusion A convenient formalism is developed, based on the determinant method, that allows one to calculate the partition function $Z^{(N)}$, Eq. (16), for the 1D Ising model with an arbitrary nearest-neighbor random exchange integral, temperature, and magnetic field. It is shown that the calculation of $Z^{(N)}$ can be reduced to solving the recurrence relation given in Eq. (15) which is convenient both for numerical as well as for analytical approaches. Particularly, we have calculated asymptotic expressions for the ensemble averaged free energy $\langle F\rangle$ and for the averaged magnetization $\langle M\rangle$ in the intermediate region, i.e., when the magnetic field (or temperature) is not very high or low. We have evaluated also the free energy when possessing random magnetic field in each site in addition to nearest-neighbor random exchange integrals. The zeros of the partition function for the Ising model in the complex external magnetic field plane coincide with the singularities of the real part of transmission amplitude through the 1D chain of δ-function potentials.

It should be clear that the concepts discussed in this article apply not only to the discrete random magnetic field h_j but can be generalized to the problem when both h_j and I_j are random and described by continuous distribution functions, e.g., Gaussian or uniform.

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