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Kramers-Kronig relations and the barrier interaction time problem

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Abstract. It is shown that the two characteristic interaction times $\tau_1(\omega)$ and $\tau_2(\omega)$ for classical electromagnetic waves with an arbitrarily shaped barrier are not independent quantities, but are connected by Kramers-Kronig relations for the real and imaginary components of a causal magnitude. The corresponding macroscopic sum rule for the complex time is also derived. An analogy between the interaction time problem and an electrical circuit with capacitive and conducting frequency dependent components is established.

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In the theory of the traversal time problem of electrons and electromagnetic waves (EMW), two characteristic times have arisen in many approaches [1–5]. Both times are the real and imaginary components of a complex quantity. We know that any experiment must measure a real quantity, and so the outcome of any measurement of the interaction time must be a real quantity, possibly involving the two characteristic times. These two times are related as a consequence of the analytical properties of the complex quantity whose real and imaginary components are the two characteristic times. In this paper we would like to study the relationship between these two characteristic times.

The concept of a complex interaction time arose in the Feynman path-integral approach [6] when Sokolovski and Baskin [7] applied this kinematic approach to quantum mechanics by a formal generalization of the classical time concept to the traversal time in a finite region. Leavens and Aers [8] reinterpreted the oscillatory amplitude approach, proposed by Büttiker and Landauer [9,10], in terms of a complex time. Jauho and Jonson [11] extended in a similar direction the time-modulated barrier approach, also proposed by Büttiker and Landauer [12]. Fertig [13] arrived at the concept of a complex time and derived the complex distribution of traversal times for a particle tunneling through a rectangular barrier. Recently, Balcou and Dutriaux [14] experimentally investigated the tunneling times associated with frustrated total internal reflection of light. They have shown that the two characteristic times correspond, respectively, to the spatial and angular shifts of the beam.

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Gasparian *et al.* have shown [15,16], with the help of the Green's function formalism, that the two characteristic times appearing in the Larmor clock approach for electrons correspond to the real and imaginary components of a single quantity, defined as an integral of the Green's function $G(x,x;E)$ for an open and finite system with length L

$$\tau = \hbar \int_0^L G(x,x;E)\mathrm{d}x = -\mathrm{i}\hbar \left[\frac{\partial \ln t}{\partial E} - \frac{r+r'}{4E}\right] \qquad (1)$$

where t is the complex amplitude of transmission, r and r' are the reflection amplitudes from the left and from the right, respectively. They also obtained [4], with the Faraday rotation scheme, a similar result to equation (1) for the characteristic interaction time of an EMW. The Faraday rotation is analogous to the magnetic clock and plays for light the same role as the Larmor precession for electrons [1,17]. The emerging EMW is elliptically polarized and the major axis of the ellipse is rotated with respect to the original direction of polarization. All relevant information about both the angle of rotation and the degree of ellipticity is contained in the complex angle θ

$$\theta = -\frac{\mathrm{i}}{2} \ln \frac{t_+}{t-} = \theta_1 - \mathrm{i}\theta_2. \qquad (2)$$

Making use of the expression for the complex amplitude of transmission $t_\pm = {T_\pm}^{1/2} \exp\{\mathrm{i}\psi_\pm\}$, one can easily check that the real part of the angle θ is equal to:

$$\theta_1 = \frac{\psi_+ - \psi_-}{2} \cdot \qquad (3)$$

This corresponds to the Faraday rotation, which results from the phase difference between left and right polarized light. The imaginary part of θ is:

$$\theta_2 = \frac{1}{4} \ln \frac{T_+}{T_-}, \tag{4}$$

and corresponds to the ratio of ellipticity [4]. It is straightforward to show, using the explicit expression of the coefficient of transmission $T_\pm$ for the dielectric slab (see, *e.g.* [18], $A_\pm = n_1/n_\pm$ and $\Delta_\pm = \omega n_\pm L/c$)

$$T_\pm = \left\{ 1 + \left(\frac{1 - A_\pm^2}{2A_\pm} \sin \Delta_\pm \right)^2 \right\}^{-1}, \tag{5}$$

and equation (4) that, depending of the values of Δ_+ and Δ_-, we may observe, during one period $[\pi m, \pi(m+1)]$ (m is integer number) linearly, circularly, and elliptically polarized light, which at πm rotates in a clockwise direction and later changes the direction of rotation.

Both effects mentioned above are quantified through the complex angle θ which depends on the time the EMW spends in the slab. This motivated us to associate a complex interaction time of the light in the region with magnetic field with this complex magnitude. We arrived at a complex characteristic interaction time τ for a EMW in the slab which can be written in terms of derivatives with respect to frequency as [4]:

$$\tau(\omega) = -\mathrm{i} \left[\frac{\partial \ln t}{\partial \omega} - \frac{r + r'}{4\omega} \right] = \tau_1(\omega) - \mathrm{i}\tau_2(\omega). \tag{6}$$

As was shown by Ruiz *et al.* [19] this is a general expression for the interaction time of an EMW with a one-dimensional region with an arbitrary index of refraction distribution, independently of the model considered. It can be rewritten in terms of the GF for photons analogously to equation (1), because all the general properties of the GF formalism for electrons which lead us to equation (1) are valid for any wave (sound or electromagnetic), whenever its propagation through a medium is described by a differential equation of second order [20].

It is the purpose of this work to discuss the properties of the EMW interaction time arising from its complex nature and to show that the frequency dependence of the real and imaginary parts of the complex interaction time are connected by the Kramers-Kronig relations. We also establish an analogy with an electrical circuit equivalent.

For the dielectric slab equation (6) leads us to the following expressions for the two time components [4]:

$$\tau_1^{\mathrm{sl}}(\omega) = \frac{T\tau_0^{\mathrm{sl}}}{2A} \left\{ \left(1 + A^2\right) + \left(1 - A^2\right) \frac{\sin 2\Delta}{2\Delta} \right\}, \tag{7}$$

and

$$\tau_2^{\mathrm{sl}}(\omega) = \frac{T\tau_0^{\mathrm{sl}}}{2A} \frac{1 - A^2}{2A} \left\{ \left(1 - A^2\right) \frac{\sin 2\Delta}{2} + \left(1 + A^2\right) \frac{\sin^2 \Delta}{\Delta} \right\}. \tag{8}$$

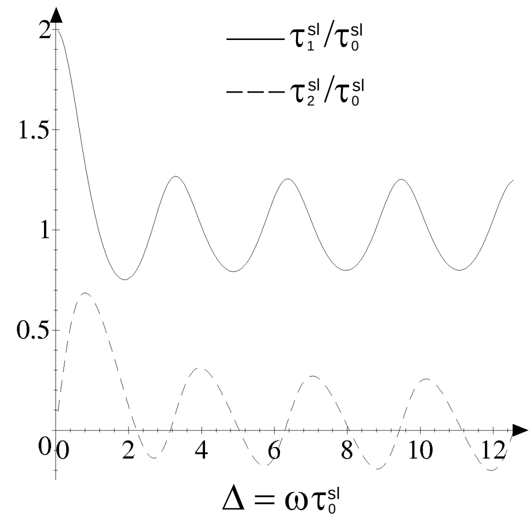

Fig. 1. The two components of the EMW interaction time $\tau_1^{\mathrm{sl}}(\omega)$ and $\tau_2^{\mathrm{sl}}(\omega)$ for a dielectric slab as a function of the incident frequency. The values of the parameters are $n_0 = 1$ and $n_1 = 2$.

where $\tau_0^{\mathrm{sl}} = L/v$ is the time that light with velocity $v = c/n_0$ would take to cross the slab, when reflection in the boundaries is not important, $\Delta = \omega\tau_0^{\mathrm{sl}}$, $A = n_1/n_0$, n_0 is the refraction index of the slab and n_1 the refraction index of the two semi-infinite media outside the slab. T is the transmission amplitude for the slab in the absence of a magnetic field and is given by equation (5) with the replacements $A_\pm \to A$ and $\Delta_\pm \to \Delta$.

Note that the first term on the RHS of equations (7, 8) which is proportional to the imaginary and real parts of $\partial \ln t/\partial\omega$ mainly contains information about the region of the slab. Most of the information about the boundary is provided by the term proportional to the reflection amplitude, r/ω, and is of the order of the wavelength λ over the length of the system L. Thus, it becomes important for low energies and/or short systems, as can be noted in Figure 1, where we present $\tau_1(\omega)$ and $\tau_2(\omega)$ given by equations (7, 8), as a function of $\Delta = \omega\tau_0$ for the first 4 periods.

$\tau_1(\omega)$ is proportional to the integrated density of states (DOS) [15,16]. It is always positive and reproduces the characteristic features of the coefficient of transmission T, *i.e.* it has a maximum at $\Delta_0 = \pi m$, where m is integer number (see Fig. 1). The sharpness and the breadth of the peaks depend on the ratio $A = n_1/n_0$. At $\Delta_1 \approx \pi/2 + \pi m$ the DOS has a minimum in accordance with equation (7). As it was pointed out in reference [21], a calculation of the DOS without taking into account the second term in equation (6) yields a wrong result without oscillation term. Such oscillations in DOS and the partial DOS should influence the conduction properties of sufficiently small conductors [22] and, as was shown in reference [23] similar correction terms in two-dimensional mesoscopic conductors are needed to obtain precise current conservation.

As for the imaginary part τ_2, we can see from Figure 1 that it is positive in the range $2\pi m \le \Delta \le \Delta_2 + 2\pi m$ and negative in the range $\Delta_2 + 2\pi m \le \Delta \le \pi + 2\pi m$, where the points Δ_2 are the solutions of the equation

$$\cos\Delta_2 + \frac{1-A^2}{1+A^2}\frac{\sin\Delta_2}{\Delta_2} = 0. \qquad (9)$$

If $r/\omega \ll 1$ then we approximately have $\Delta_2 \approx \pi/2$ (from now on we will consider only the interval $[0,\pi]$). In the same limit, $\tau_2^{\rm sl}(\omega)$ reaches both its maximum and minimum values. Its maximum is equal to

$$\tau_2^{\rm sl}(\omega)\big|_{\max} = \frac{\tau_0}{2}\left(\frac{1-A^2}{1+A^2}\right)^2 \qquad (10)$$

and occurs at $\Delta_3 = \arctan 2A/(1+A^2)$. The minimum of $\tau_2(\omega)$ appears at $\pi - \Delta_3$:

$$\tau_2^{\rm sl}(\omega)\big|_{\min} = -\frac{\tau_0}{2}\left(\frac{1-A^2}{1+A^2}\right)^2. \qquad (11)$$

We would like to emphasize that the change in sign of τ_2 (Fig. 1) is just related to a change in the direction of rotation from clockwise to counterclockwise of the transmitted wave.

Note that the $\tau_2^{\rm sl}(\omega)$ for EMW interaction time with a dielectric slab is similar to Büttiker's time (in the Larmor clock approach), which tends to align the spins parallel to the magnetic field in order to minimize its energy [1]. For energies E of incident electrons larger than the height V of a rectangular barrier, Büttiker's time changes its sign from positive to negative and so on, which is only due to the fact that the direction of the spins in the transmitted wave can be parallel or antiparallel to the magnetic field.

It is known that the frequency dependence of the real and imaginary parts of certain complex physical quantities are interrelated by the Kramers-Kronig relations, *e.g.*, the real (dispersive) part of the complex dielectric function $\epsilon(\omega)$ to its imaginary (dissipative) part, the frequency dependent real and imaginary parts of an electrical impedance, etc. [24]. The derivation of these relations is based on the fulfillment of four general conditions of the system: causality, linearity, stability and that the value of the physical quantity considered is assumed to be finite at all frequencies, including $\omega \to 0$ and $\omega \to \infty$. If these four conditions are satisfied, the derivation of Kramers-Kronig relations is purely a mathematical operation which does not reflect any other physical properties or conditions of the system. These integral relations are very general and have been used in the theory of classical electrodynamics, particle physics and solid state physics as well as in the analysis of electrical circuits and electrochemical systems (see, *e.g.* [25]).

As Thouless has shown [26], a dispersion relation exists between the length of localization and the DOS which was rewritten in reference [27] in the form of linear dispersion relations between the real and imaginary parts of $\ln t$, *i.e.*, the logarithm of the complex transmission amplitude. Using this dispersion relation it is straightforward to show that the complex interaction time, $\tau(\omega)$ (6), is an analytical function of frequency in the upper half of the complex ω-plane (see *e.g.* [24]). In other words the four conditions mentioned above are fulfilled for the complex time (6) and the following relationship between the $\tau(\omega)$ and its complex conjugate $\tau^*(\omega)$ holds on the real axis (see Eq. (6)):

$$\tau(\omega) = \tau^*(-\omega) \qquad (12)$$

which means that the complex interaction time $\tau(\omega)$ has the following properties:

$$\tau_1(\omega) = \tau_1(-\omega),\ \tau_2(\omega) = -\tau_2(-\omega). \qquad (13)$$

Therefore, the real part $\tau_1(\omega)$ is an even function of frequency and can have a finite value at zero frequency (for the slab we have $\tau_1^{\rm sl}(0) = L/vA$). As for the imaginary part $\tau_2(\omega)$, it is an odd function and must vanish in the limit of zero frequency: $\tau_2(0) = 0$. These conditions imply that the real and imaginary components of the time likewise obey Kramers-Kronig integral relations, and so we may write

$$\tau_1(\omega) = \tau_0 + \frac{2}{\pi}{\rm P}\int_0^\infty \frac{y\tau_2(y)}{y^2-\omega^2}{\rm d}y \qquad (14)$$

$$\tau_2(\omega) = -\frac{2\omega}{\pi}{\rm P}\int_0^\infty \frac{\tau_1(y)-\tau_0}{y^2-\omega^2}{\rm d}y \qquad (15)$$

where P means principal part and $\tau_0 = Ln/c$, *i.e.* the crossing time in the dielectric system, without any boundary.

In particular, if we make $\omega = 0$ in equation (15), we arrive at so called "macroscopic sum rule" for the complex interaction time

$$\tau_1(0) = \tau_0 + \frac{2}{\pi}\int_0^\infty \frac{\tau_2(y)}{y}{\rm d}y \qquad (16)$$

Thus we see from equation (16) that in general if no imaginary component $\tau_2(\omega)$ exists at any frequency, then $\tau_1 = \tau_0$ always holds. In the case of the interaction time in the dielectric slab, the integral relations (14–16), which are the central result of our work, can be verified, using the explicit expressions (7, 8) (see, *e.g.*, [28]).

Note that the validity of the Kramers-Kronig relations for the complex interaction time has a rather deep significance because it may be demonstrated that these conditions are a direct result of the causal nature of physical systems by which the response to a stimulus never precedes the stimulus. At this point it is worth mentioning that the experiments with, *e.g.*, undersized waveguides [29,30] or periodic dielectric heterostrucures [31,32], where the so called "superluminal velocities" have been observed for the barrier tunneling time need to be interpreted carefully.

It is known that for the description of the Debye relaxation behavior it is possible to replace the complex dielectrics by an electric circuit equivalent. Because of the close resemblance to this class of linear response physical phenomena, it is possible to map the interaction time problem by a circuit equivalent consisting of a frequency dependent capacitance $C(\omega)$ and a frequency dependent

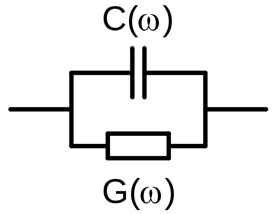

Fig. 2. Schematic representation of the equivalent circuit with capacitive $C(\omega)$ and conducting $G(\omega)$ frequency dependent components for the EMW interaction time.

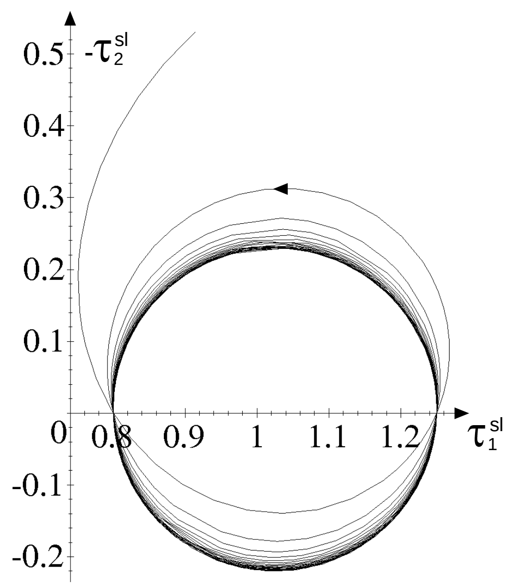

Fig. 3. Complex plane interaction time diagram for a dielectric slab. The arrow indicates the direction of increasing frequency.

conductance $G(\omega)$, as shown in Figure 2. (see, *e.g.* [25]). This means that the natural way of describing the barrier interaction time problem is *via* two parallel channels which correspond to two mechanisms for the same physical phenomenon.

Let us represent the complex time components $\tau_1^{sl}(\omega)$, equation (7), and $\tau_2^{sl}(\omega)$, equation (8), in the complex plane. They are plotted against one another in Figure 3. We see that for small frequencies we have a skewed arc. With increasing frequency, the influence of the second terms in equations (7, 8), due to boundary effects, becomes less important and the curve, in the limit $\omega \to \infty$ approximates to an ideal circle.

Note that in the case of the Debye dispersion relations for the complex dielectric function $\epsilon(\omega)$, an ideal semicircle in the complex plane means that we deal with a single relaxation time. In our case it means that for high frequency/or short wavelength we deal with the classical crossing time, taking into account multiple reflection in the slab [19]. It is not difficult to show that in a such limit we have

$$\left(\tau_2^{sl}\right)^2 + \left\{\tau_1^{sl} - \left[\frac{\tau_0^{sl}}{2A}\left(1+A^2\right) - r\right]\right\}^2 = r^2 \qquad (17)$$

which is the equation of a circle in the complex plane of $-\tau_2^{sl}$ and τ_1^{sl} with the centre $\left\{\tau_0^{sl}/2A\left(1+A^2\right) - r, 0\right\}$ and with a radius given by

$$r = \frac{\tau_0^{sl}}{4A}\frac{\left(1-A^2\right)^2}{1+A^2}\,. \qquad (18)$$

We showed that the two components, the real part $\tau_1(\omega)$ and the imaginary part $\tau_2(\omega)$, of the complex barrier interaction time for EMW are not entirely independent quantities, but connected by Kramers-Kronig relations.

The barrier interaction time problem for EMW in a slab can be mapped to a circuit equivalent consisting of a parallel combination of a frequency dependent capacitance $C(\omega)$ and a frequency dependent conductance $G(\omega)$ in a network. There are two distinct times: the first one due to the propagation of the EMW in the real (dispersive) part, ϵ_1 of the complex dielectric function ϵ, and the second one due to the imaginary (dissipative) part, ϵ_2 of the complex dielectric function ϵ. Note that in this paper the validity of the Kramers-Kronig relations was only checked analytically for EMW, but in general this implies that they are also valid for all quantum particles represented by a differential equation of second order as indicated by the numerical calculations for the complex tunneling time for electrons.

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References

1. M. Büttiker, Phys. Rev. B **27**, 6178 (1983).
2. E.H. Hauge, J.A. Støvneng, Rev. Mod. Phys. **61**, 917 (1989).
3. R. Landauer, Th. Martin, Rev. Mod. Phys. **66**, 217 (1994).
4. V. Gasparian, M. Ortuño, J. Ruiz, E. Cuevas, Phys. Rev. Lett. **75**, 2312 (1995).
5. Th. Martin, Int. J. Mod. Phys. B **10**, 3747 (1996).
6. R.P. Feynman, A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
7. D. Sokolovski, L.M. Baskin, Phys. Rev. A **36**, 4604 (1987).
8. C.R. Leavens, G.C. Aers, Solid State Commun. **63**, 1101 (1987).
9. M. Büttiker, R. Landauer, Phys. Scripta **32**, 429 (1985).
10. M. Büttiker, R. Landauer, IBM J. Res. Develop. **30**, 451 (1986).
11. A.P. Jauho, M. Jonson, J. Phys.-Cond. **1**, 9027 (1989).
12. M. Büttiker, R. Landauer, Phys. Rev. Lett. **49**, 1739 (1982).
13. H.A. Fertig, Phys. Rev. Lett. **65**, 2321 (1990).
14. Ph. Balcou, L. Dutriaux, Phys. Rev. Lett. **78**, 851 (1997).
15. V. Gasparian, M. Pollak, Phys. Rev. B **47**, 2038 (1993).

16. V. Gasparian, M. Ortuño, J. Ruiz, E. Cuevas, M. Pollak, Phys. Rev. B **51**, 6473 (1995).
17. I.A. Baz', Sov. J. Nucl. Phys. **4**, 182 (1967).
18. J.K.D. Möller, *Optics* (University Science Books, California, 1988).
19. J. Ruiz, M. Ortuño, E. Cuevas, V. Gasparian, J. Phys. I France **7**, 653 (1997).
20. A.G. Aronov, V.M. Gasparian, U. Gummich, J. Phys.-Cond. **3**, 3023 (1991).
21. V. Gasparian, T. Christen, M. Büttiker, Phys. Rev. A **54**, 4022 (1996).
22. M. Büttiker, Phys. Rev. B **40**, 3405 (1989).
23. J. Wang, Q. Zeng, H. Gou, Phys. Rev. B **55**, 9770 (1989).
24. L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon Press, New York, 1982).
25. J.R. Macdonald, *Impedance Spectroscopy* (John Wiley and Sons, New York, 1987).
26. D.J. Thouless, Phys. Rep. **136**, 94 (1974).
27. V.M. Gasparian, B.L. Altshuler, A.G. Aronov, Z.H. Kasamanian, Phys. Lett. A **132**, 201 (1988).
28. I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1965).
29. A. Enders, G. Nimtz, J. Phys. I France **2**, 1693 (1992).
30. A. Enders, G. Nimtz, Phys. Rev. E **48**, 632 (1993).
31. A.M. Steinberg, P.G. Kwiat, R.Y. Chiao, Phys. Rev. Lett. **71**, 708 (1993).
32. Ch. Spielman, R. Szipöcs, A. Stingl, F. Krausz, Phys. Rev. Lett. **73**, 708 (1994).