Ann. Phys. (Leipzig) 7 (1998) 7 — 8, 756 — 763

On the application of the Kramers-Kronig relations to the interaction time problem

V. Gasparian[1], **G. Schön**[2], **J. Ruiz**[3], **M. Ortuño**[3]

[1]Department of Physics, Yerevan State University, 375049 Yerevan, Armenia
[2]Institut für Anorganische Chemie, Universität–GH Essen, D-45117 Essen, Germany
[3]Departamento de Física, Universidad de Murcia, Spain

Received 29 September 1998, accepted 20 October by P. Mittelstaedt

Abstract. It is shown that the two components of the complex characteristic interaction time $\tau(\omega) = \tau_1(\omega) - i\tau_2(\omega)$ for classical electromagnetic waves with an arbitrary shaped barrier are not entirely independent quantities, but are connected by the Kramers-Kronig relations. The corresponding macroscopic sum rule for the complex time is also derived. An analogy between the interaction time problem and an electrical circuit with capacitive and conducting components is established from which we propose that the effective crossing time should be the maximum of the two components.

1 Introduction

Although common sense dictates that the tunneling time must be a real quantity and that there are no clocks that measure a complex time, nevertheless the concept of complex time in the theory of the traversal time problem of electrons and electromagnetic waves (EMW) has arisen in many approaches [1, 2, 3, 4]. Pollak and Miller [5], while studying the average tunneling time in classical chemical systems, arrived at the concept of an imaginary time through the flux-flux correlation function. A few years later, the concept of a complex time arose more naturally in the Feynman path–integral approach [6], when Sokolovski and Baskin [7] applied this kinematic approach to quantum mechanics by a formal generalization of the classical time concept $\tau^{\rm cl}$ to the traversal time in a finite region. The oscillatory amplitude approach was proposed by Büttiker and Landauer [8, 9]. When applied without resorting to the WKB approximation, it led Leavens and Aers [10] to complex times too. Jauho and Jonson [11] showed that a general analysis of the time-modulated barrier approach, which was proposed by Landauer and Büttiker [12], leads again to a complex quantity. Fertig [13] arrived at the same concept of complex time and derived the complex distribution of traversal times for a particle tunneling through a rectangular barrier (see also Ref. [14]). Recently Martin [4] provided a general framework for the formulation of the traversal time by using the Feynman path integral approach. This formulation of the problem leads to the same complex time as one would have expected. Gasparian *et al.* [15, 16] have shown, with the help of the Green's function formalism, that the two characteristic times appearing in the Larmor clock approach for electrons correspond to the real and imaginary components of a single quantity defined as an integral of the Green's function $G(x,x;E)$ for an open and finite system with length L

$$\tau = \hbar \int_0^L G(x,x;E)\,dx = -i\hbar \left[\frac{\partial \ln t}{\partial E} - \frac{r+r'}{4E}\right], \tag{1}$$

where t is the complex amplitude of transmission and r and r' are the reflection amplitudes from the left and from the right, respectively. For a spatially symmetric barrier $V(x) = V(-x)$ one additionally has $r = r'$. In Ref. [3], with the Faraday rotation scheme, a very similar result to Eq. (1) was obtained for the characteristic interaction time τ of an EMW. In the finite system, which represents our magnetic clock, the Faraday rotation plays for light a similar role as the Larmor precession for electrons [17, 18]. The emerging EMW is elliptically polarized and the major axis of the ellipse is rotated with respect to the original direction of polarization. Both effects are quantified through the complex angle θ which depends on the time the EMW spends in the slab. This motivated us to associate with this complex magnitude a complex interaction time for the light in the region with magnetic field [3, 19]. Recently, Balcou and Dutriaux [20] experimentally investigated the tunneling times associated with frustrated total internal reflection of light. They have shown that the real and imaginary parts of the complex tunneling time correspond, respectively, to the spatial and angular shifts of the beam. Note that in most tunneling experiments, instead of electrons, electromagnetic waves were used to exclude interaction effects (see, e.g. [21, 22, 23, 24]). The paper is organized as follows. In the next section we obtain the general properties of the two components of the barrier interaction time in a slab. In Sec. 3 we deduce the Kramers-Kronig relations for the time components and establish an analogy with an electrical circuit. In Sec. 4 we study the relation between the two time components for periodic systems.

2 Results for a slab

For the dielectric slab, Eq. (1) leads us to the following expressions for the two time components [3]

$$\tau_1^{\rm sl}(\omega) = \frac{T\tau_0^{\rm sl}}{2A}\left\{\left(1+A^2\right) + \left(1-A^2\right)\frac{\sin 2\Delta}{2\Delta}\right\}, \tag{2}$$

and

$$\tau_2^{\rm sl}(\omega) = \frac{T\tau_0^{\rm sl}}{2A}\frac{1-A^2}{2A}\left\{\left(1-A^2\right)\frac{\sin 2\Delta}{2} + \left(1+A^2\right)\frac{\sin^2\Delta}{\Delta}\right\}, \tag{3}$$

where $\tau_0^{\rm sl} = L/v$ is the time that light with velocity $v = c/n_0$ would take to cross the slab when reflection in the boundaries is not important. Moreover, $\Delta = \omega\tau_0^{\rm sl}$ and $A = n_1/n_0$, where n_0 is the refractive index of the slab and n_1 the refractive index of the two semi-infinite media outside the slab. T is the transmission amplitude for the slab in the absence of a magnetic field and is given by

$$T = \left\{1 + \left(\frac{1-A^2}{2A}\sin\Delta\right)^2\right\}^{-1}. \tag{4}$$

Note that the first term on the r.h.s. of Eqs. (2) and (3), which is proportional to the imaginary and real parts of $\partial \ln t/\partial\omega$ respectively, mainly contains information about the region of the slab. Most of the information about the boundary is provided by the term proportional to the reflection amplitude, r/ω, which is of the order of the wavelength λ over the length of the system L. Thus, it becomes important for low energies and/or short systems. The time $\tau_1(\omega)$ is proportional to the integrated density of states (DOS) [15, 16]. It is always positive and reproduces the characteristic features of the coefficient of transmission T, i.e., it has a maximum at $\Delta_0 = \pi m$, where m is an integer number. The sharpness and the breadth of the peaks depend on the ratio $A = n_1/n_0$. At $\Delta_1 \approx \pi/2 + \pi m$, the DOS has a minimum in accordance with Eq. (2). As it was pointed out in Ref. [25], a calculation of the DOS without taking into account the second term in Eq. (1) yields a wrong result without oscillation terms. Such oscillations in the DOS and the partial DOS should influence the conduction properties of sufficiently small conductors [26] and, as was shown in Ref. [27], similar correction terms in two-dimensional mesoscopic conductors are needed to obtain precise current conservation. Before closing this section, let us note that from the discussions above it follows that the two components $\tau_1^{\rm sl}(\omega)$ and $\tau_2^{\rm sl}(\omega)$ of the complex time are not independent quantities, but are connected through Kramers-Kronig relations, as we shall now show.

3 Kramers-Kronig relations

The real and imaginary parts of certain complex physical quantities are interrelated by Kramers–Kronig relations, e.g., the real (dispersive) part of the complex dielectric function $\epsilon(\omega)$ and its imaginary (dissipative) part or the frequency dependent real and imaginary parts of an electrical impedance, etc. [28]. The derivation of these relations is based on the fulfillment of four general conditions: causality, linearity, stability and that the value of the physical quantity considered is assumed to be finite at all frequencies, including $\omega \to 0$ and $\omega \to \infty$. If these four conditions are satisfied, the derivation of Kramers–Kronig relations is a direct, purely mathematical operation. These integral relations are very general and have been used in the theory of classical electrodynamics, particle physics and solid state physics as well as in the analysis of electrical circuits and electrochemical systems, see, e.g., [29]. A dispersion relation between the localization length and the DOS was obtained by Thouless [30] and rewritten in Ref. [31] in the form of a linear dispersion relation between the real and imaginary parts of the logarithm of the complex transmission amplitude. From this dispersion relation it is straightforward to show that the complex interaction time is an analytical function of frequency in the upper half of the complex ω-plane, see, e.g., [28]. The four conditions mentioned above are fulfilled for the complex time (1) and the following relationship between the $\tau(\omega)$ and its complex conjugate $\tau^*(\omega)$ holds on the real axis (see Eq. (1)):

$$\tau(\omega) = \tau^*(-\omega). \tag{5}$$

Therefore, the real part $\tau_1(\omega)$ is an even function of frequency and can have a finite value at zero frequency (for the slab we have $\tau_1^{\rm sl}(0) = L/vA$). On the contrary, the imaginary part $\tau_2(\omega)$ is an odd function of frequency and must vanish in the limit of zero frequency. These conditions imply that the real and imaginary components of the time obey Kramers-Kronig integral relations. Hence the real component can be writen as

$$\tau_1(\omega) = \tau_0 + \frac{2}{\pi}\mathbf{P}\int_0^\infty \frac{y\tau_2(y)}{y^2-\omega^2}dy, \tag{6}$$

while the imaginary component can be expressed as

$$\tau_2(\omega) = -\frac{2\omega}{\pi}\mathbf{P}\int_0^\infty \frac{\tau_1(y)-\tau_0}{y^2-\omega^2}dy, \tag{7}$$

where $\mathbf{P}$ means principal part and τ_0 is the crossing time in the dielectric system, assuming no boundaries. From Eq. (6) we deduce, provided the imaginary time component $\tau_2(\omega)$ is zero at all frequencies, that $\tau_1 = \tau_0$ always holds. For the dielectric slab, the integral relations (6) and (7), which are the central result of our work, can be explicitly verified by using the expressions (2) and (3) (see, e.g., [32]). The validity of the Kramers–Kronig relations for the complex interaction time has a rather deep significance because they are a direct result of the causal nature of physical systems by which the response to a stimulus never precedes the stimulus. They can also serve as a starting point for the understanding of the origin of the complex time and state that the interaction time for any classical or quantum-mechanical wave will always have two components. At this point it is worth mentioning that the experiments with, e.g., undersized waveguides [21, 22] or periodic dielectric heterostrucures [23, 24], where the so called "superluminal velocities" have been observed for the barrier tunneling time, need to be interpreted carefully. The analogy between the complex time and the complex dielectric function can serve to map the interaction time problem to a two channel electrical circuit. For the purpose of illustration, let us expand the expressions (2) and (3) for $\tau_1^{\rm sl}(\omega)$ and $\tau_2^{\rm sl}(\omega)$ near $\omega \approx 0$,

$$\tau_1^{\rm sl}(\omega) = \frac{\tau_0}{A}\left\{1+\left(\frac{1-A^2}{2A}\,\omega\tau_0\right)^2\right\}^{-1}, \tag{8}$$

and

$$\tau_2^{\rm sl}(\omega) = \frac{\tau_0\left(1-A^2\right)}{A^2}\left\{1+\left(\frac{1-A^2}{2A}\,\omega\tau_0\right)^2\right\}^{-1}. \tag{9}$$

The dependences of $\tau_1^{\rm sl}$ and $\tau_2^{\rm sl}$ on frequency, apart from some irrelevant factors, are similar to the Debye expressions for the real and imaginary parts of the complex dielectric function $\epsilon^*(\omega) - \epsilon_\infty$ (see, e.g., [29]),

$$\epsilon_1 - \epsilon_\infty = \frac{\epsilon_s - \epsilon_\infty}{1+(\omega\tau)^2}, \tag{10}$$

and

$$\epsilon_2 = \frac{(\epsilon_s - \epsilon_\infty)\,\omega\tau}{1+(\omega\tau)^2}, \tag{11}$$

where ϵ_s and ϵ_∞ are the static and the high–frequency dielectric constants, respectively. This analogy, apart from its direct relevance to a class of linear response phenomena, makes it possible to replace the dielectric slab in a magnetic field, i.e. our magnetic clock for the EMW traversal time, by a circuit analog which consists of a parallel combination of a frequency dependent capacitance $C(\omega)$ and a frequency dependent conductance $G(\omega)$. Thus the natural way of describing the barrier interaction time problem is via two parallel channels which correspond to two mechanisms of similar physical phenomena. In other words, $\tau_1(\omega)$ and $\tau_2(\omega)$ are the times spent by the EMW in the capacitive and conducting channels, respectively.

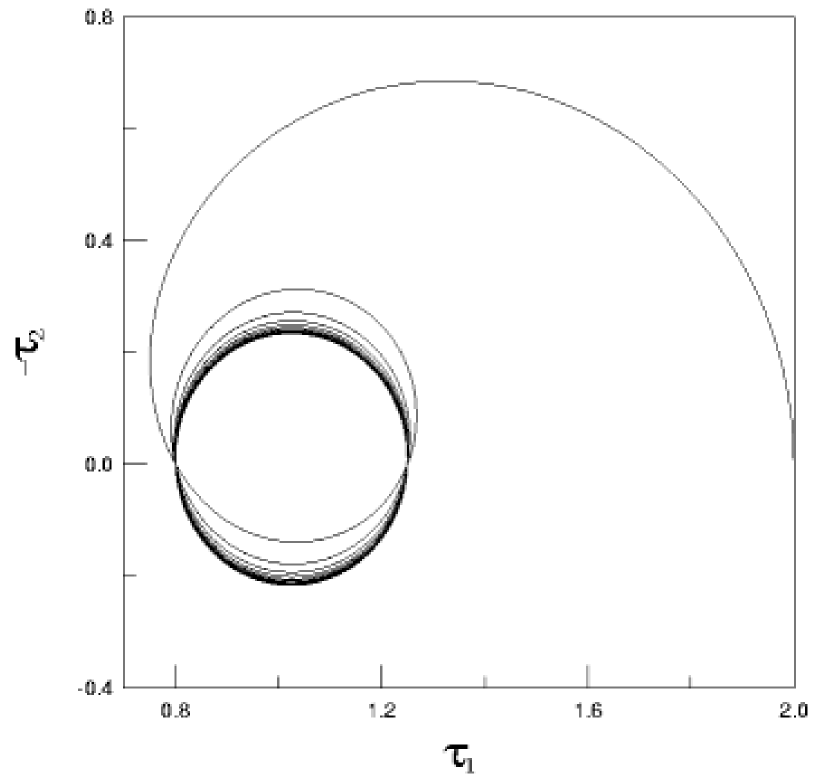

Fig. 1 Interaction time diagram for a dielectric slab in the complex plane: $-\tau_2^{\mathrm{sl}}(\omega)$ is plotted vs $\tau_1^{\mathrm{sl}}(\omega)$ for each frequency. With increasing frequencies the curve tends towards a perfect circle.

In Fig. 1 we plotted the complex time components $-\tau_2^{\mathrm{sl}}(\omega)$, Eq. (2), versus $\tau_1^{\mathrm{sl}}(\omega)$, Eq. (3), in units of τ_0. Each point corresponds to a frequency, and with increasing frequencies the curve tends towards a perfect circle. We see that for small frequencies we have a skewed arc. With increasing frequency, the influence of the second term in Eqs. (2) and (3), due to boundary effects, becomes less important and the curve, in the limit $\omega \to \infty$, approaches an ideal circle of radius

$$r = \frac{\tau_0^{\mathrm{sl}}}{4A} \frac{\left(1 - A^2\right)^2}{1 + A^2}. \tag{12}$$

4 Results for a periodic system

We now consider a periodic arrangement of layers. Layers with refractive index n_1 and thickness d_1 alternate with layers of refractive index n_2 and thickness d_2. The wavenumbers in the layers of the first and second type are $k_1 = \omega n_1/c$ and $k_2 = \omega n_2/c$, respectivley, and $a = d_1 + d_2$ is the spatial period. The periodicity of the

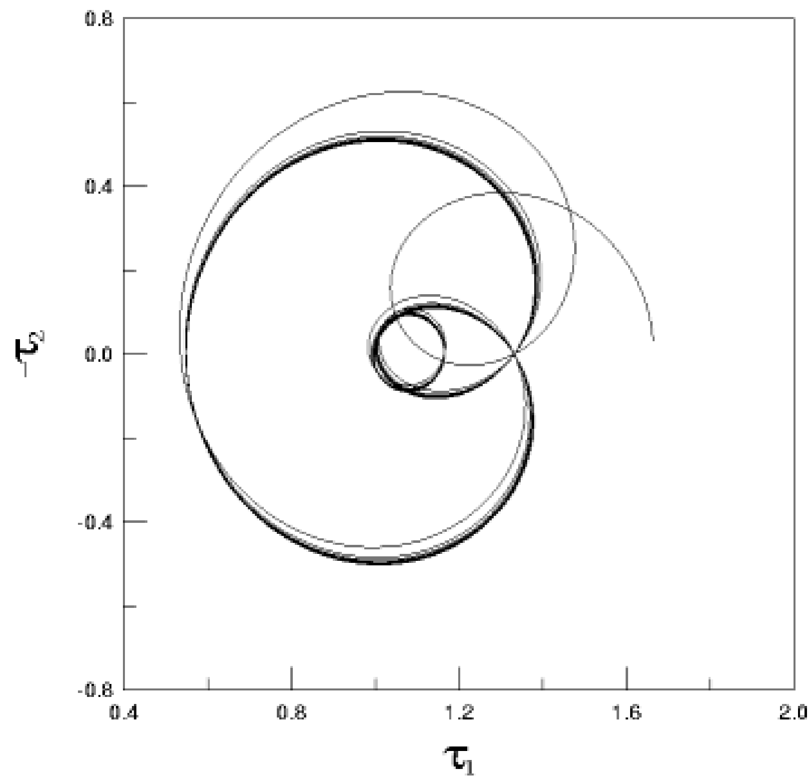

Fig. 2 The time $-\tau_2(\omega)$ as a function of $\tau_1(\omega)$ for a double barrier system.

system enables us to obtain analytically the transmission amplitude by using the characteristic determinant method [19]:

$$t_N = \frac{e^{-ik_1 d_1}}{\cos\frac{N\beta a}{2} - i\frac{\sin\frac{N\beta a}{2}}{\sin\beta a}\sqrt{\sin^2(\beta a) + \left[\frac{k_1^2 - k_2^2}{2k_1 k_2}\sin(k_2 d_2)\right]^2}}\,, \tag{13}$$

where β plays the role of quasimomentum of the system and is defined by

$$\cos(\beta a) = \cos(k_1 d_1)\cos(k_2 d_2) - \frac{k_1^2 + k_2^2}{2k_1 k_2}\sin(k_1 d_1)\sinh(k_2 d_2)\,. \tag{14}$$

When the modulus of the r.h.s. of Eq. (14) is greater than 1, β has to be taken as imaginary. This situation corresponds to a forbidden frequency window. The two components of the interaction time are obtained by substituting the previous value of the transmission amplitude, Eq. (13), in the expression for the complex time Eq. (1). We concentrate on the simplest periodic case with the choice $n_1 d_1 = n_2 d_2$, which is used in most experimental setups [23]. We consider systems with alternating refractive indices of 2 and 1 and widths of 0.6 and 1.2, respectively. The simplest of these systems is the double barrier (two dielectric slabs) whose interaction time diagram is shown in Fig. 2. The times $\tau_2(\omega)$ and $\tau_1(\omega)$ are measured in units of the τ_0 of the corresponding system. At high frequencies, the curve tends to an asymptotic figure presenting a double periodicity.

5 Conclusion

We have shown that the two components of the complex barrier interaction time for EMW are not independent quantities, but are connected by Kramers–Kronig relations. Thus the response to a stimulus never precedes the stimulus. In this paper, the validity of the Kramers–Kronig relations was only checked analytically for EMW, but we believe that they are also valid for any wave governed by a differential equation of second order, as indicated by the numerical calculations for the complex tunneling time for electrons.

G.S. and V.G. would like to acknowledge financial support by the *Bundesminister für Bildung, Wissenschaft und Forschung (BMBF)* of Germany, contract number 03N1012A7, and to thank M. Büttiker, C.R. Leavens, J.G. Muga, U. Simon and H. Wiggers for helpful and stimulating discussions. J.R. and M.O. would like to acknowledge the Spanish DGES for financial support: project number PB96/1118.

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