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RESISTANCE OF A ONE-DIMENSIONAL FINITE CHAIN OF PERIODICALLY ARRANGED RANDOM SCATTERERS

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The electron resistance ρ of a one-dimensional chain of random scatterers located at the same distance to each other is considered. The localization length for uniformly distributed amplitudes of the delta-function was analytically calculated without the assumption of weak scattering. It is shown that Landauer's resistance of chain increases not exponentially but, generally speaking, by a "power" law with increasing chain length for an electron energy corresponding to the center of the zone. In the weak disordered limit the well-known relation $\ln\langle\rho\rangle = 2\langle\ln\rho\rangle$ takes place. In the other limiting case of strong scattering this relation does not take place.

THE LANDAUER'S average dimensionless resistance $\langle\rho\rangle$ of a one-dimensional (1D) metal with static disorder, where all electron states are localized, in the case of increasing sample length L at zero temperature, is usually expressed in the form ($\hbar = e^2 = 1$) [1–5]

$$\langle\rho\rangle = \tfrac{1}{2}(e^{L/\xi} - 1). \tag{1}$$

Here ξ is the localization length of the located states, which depends on the potential form inside the 1D-system and does not depend on the length and $\langle\ldots\rangle$ means ensemble averaging.

The calculation of the dependence ξ on energy E is a difficult problem by itself, hence it can be implemented for simple models only. In the case of weak electron scattering for the "white-noise"-type potential, the localization radius is calculated without taking into account an external electric field (see, e.g. [6]) and in the presence of an external homogeneous electric field as well [7]. In the case of strong scattering the localization radius is calculated in [8] by means of a new "determinant" method [9], which permits one to see what happens with electron states at increasing interaction force between the electron and scatterers with or without an external electric field.

The aim of this paper is to analytically calculate the averaging over an ensemble the resistance $\langle\rho\rangle$ for an exactly solvable disordered 1D model. Up to now this problem was solved on a computer only [8,10–12].

An explicit expression for $\xi(E)$ is obtained without the assumption of weak scattering and also outside the short-wave-length approximation $kl \gg 1$ (were k is the electron momentum and l is the mean free path). In that case it occurs that in the exactly solvable model under consideration, along with the growth of the system length, its resistance increases not exponentially but generally speaking, by a low power.

Let us consider a model in which delta-function potentials of arbitrary amplitude V_n are located periodically at $x = na$ of a chain (a is the period of the structure):

$$V(x) = \sum_{n=1}^{N} V_n \delta(x - na); \quad n = 1, 2, \ldots, N. \tag{2}$$

The solutions of the Schrodinger equation for scattering from the left, i.e. $x < x_1$ of a wave of energy $E = k^2(\hbar = 2m_0 = 1$; and m_0 is the free electron mass), may be written outside the interaction range in the usual form

$$\psi(k,x) = e^{ikx} + r(k)e^{-ikx}, \quad x \le x_1,$$

and

$$\psi(k,x) = t(k)e^{ikx}, \quad x \ge x_N.$$

Here $r(k)$ and $t(k)$ are respectively the reflection and transmission amplitudes. As it has been shown in [9, 13] the coefficient transmission T may be written as

$$T = |t(k)|^2 = |D_N|^{-2} \tag{3}$$

and the well-known Landauer's resistance ρ as

$$\rho = T^{-1} - 1 = |D_N|^2 - 1, \tag{4}$$

where D_N is the determinant of the matrix

$$(D_N)_{np} = \delta_{ne} + \frac{iV_n}{2k}\exp\{ik|x_n - x_e|\}. \tag{5}$$

The determinant D_N of the matrix (5) satisfies the recurrence equations:

$$D_N = A_N D_{N-1} - B_N D_{N-2}, \tag{6}$$

where

$$A_N = 1 + B_N + \frac{iV_N}{2k}[1 - \exp(2ika)], \quad N > 1$$

$$A_1 = 1 + \frac{iV_1}{2k}; \quad B_N = \frac{V_N}{V_{N-1}}\exp(2ika);$$

$$D_0 = 1, \quad D_{-1} = 0$$

and $D_{N-1}(D_{N-2})$ is the determinant with Nth [and $(N-1)$th] column and row removed.

As shown in [9, 13] the analytical expression for the determinant D_N was calculated in some particular cases: in the Kroning–Penny like model, in the case of strong scattering and when a resonance transmission takes place.

Note that the recurrence relationship (6) can be used conveniently in the numerical solution of the problem with an arbitrary degree of vertical disorder of a chain without taking into account the external electric field as well as in the presence of the latter [8].

After some tedious but straightforward calculations, using the recurrence relationship (6) for D_N (5) we obtain

$$D_N = 1 + \sum_{p=1}^{N} \sum_{1=j_1<\cdots<j_p}^{N} f_p(x_{j_1}, \ldots, x_{j_p}) \prod_{l=1}^{p} \frac{iV_{j_l}}{2k}. \tag{7}$$

Here phase factor f_p is a determinant of $p \times p$ matrix and matrix element has the form

$$(f_p)_{nl} = \exp ik|x_n - x_l|; \quad f_1 = 1$$

and

$$f_p = \prod_{l=1}^{p-1}[1 - \exp 2ik(x_{j_{l+1}} - x_{j_l})]. \tag{8}$$

After the substituting of expression for f_p in equation (7) the determinant D_N of this chain has the form

$$D_N = 1 + \sum_{p=1}^{N} \sum_{1=j_1<\cdots<j_p}^{N} \frac{iV_{j_1}}{2k} \cdots \frac{iV_{j_p}}{2k} \times \prod_{l=1}^{p-1}(1 - e^{2ik(x_{j_{l+1}} - x_{j_l})}). \tag{9}$$

Let us proceed to calculate the resistance of a chain with vertical disorder averaging over an ensemble. We shall assume that the amplitudes of the delta-function potential V_n where distributed uniformly in an interval $[-W/2, W/2]$. Averaging equation (4) upon the realization of a random potential distributed uniformly within a finite interval of width W we obtain the following expression for $\langle\rho\rangle$

$$\langle\rho\rangle = \sum_{p=1}^{N} \alpha^p 2^{(p-1)} \sum_{1=j_1<\ldots<j_p}^{N} \times \prod_{l=1}^{p-1}[1 - \cos 2ka(j_{l+1} - j_l)]. \tag{10}$$

Here

$$\alpha \equiv W^{-1} \int_{-w/2}^{w/2} \frac{V_j^2}{4k^2} \mathrm{d}V_j = \frac{W^2}{48k^2}.$$

It follows from equation (10) that for arbitrary energy E of an incident electron and at $\alpha \ll 1$, $\langle\rho\rangle$ has the form

$$\langle\rho\rangle \simeq N\alpha + \alpha^2\left[N^2 - 1 + \frac{\sin(N+1)ka \sin(N-1)ka}{\sin^2 ka}\right]. \tag{11}$$

If $ka = \pi n$ (see [9], i.e. in the resonance case, we find from equation (11) that

$$\langle\rho\rangle \simeq N\alpha$$

and the resistance is finite and the mean free path is proportional to α^{-1}. This result reaffirmed the result of [4], that in the short-sample limit $L \ll a\alpha^{-1}$ the average resistance $\langle\rho\rangle$ has a linear length dependence.

For the energy E of an incident electron, which corresponds to the centre of the zone, i.e. at $k\alpha = \pi/2$ we shall finally obtain from equation (10) the following compact expression for the

$$\langle\rho\rangle = \frac{1}{2}\left(\operatorname{sh} Nx + \frac{\operatorname{ch} Nx}{\operatorname{ch} x} - 1\right) \quad \text{at } N = 2m + 1, \tag{12a}$$

$$\langle\rho\rangle = \frac{1}{2}\left(\operatorname{ch} Nx + \frac{\operatorname{sh} Nx}{\operatorname{ch} x} - 1\right) \quad \text{at } N = 2m, \tag{12b}$$

where $x = 2\alpha$.

Equation (12) is the main result of our paper. It is necessary to note that equation (12) as received without the use of the perturbation theory, and it is valid for different magnitudes of the scatter of the amplitudes W and for arbitrary number of the scatters N. It can be seen from equation (12) that the mean electrical resistance of a strictly 1D system for an electron energy, corresponding to the center of the zone, has a power dependence on $N = la^{-1}$.

Let us consider the limit cases. At $\alpha \gg 1$ from equation (10) we find

$$\langle \rho \rangle \simeq \alpha^N 2^{2(N-1)} \tag{13}$$

or

$$\ln \langle \rho \rangle \simeq N \ln 4\alpha.$$

The condition $\alpha \gg 1$ means that $a/\ln 4d < a$ [see equation (17)] i.e the localization radius is smaller than the lattice constant. This case is of certain interest in the 1D superlatice, where a becomes the superlattice constant. Then each transition act through the potential barriers may be considered as an independent one, and in this case the obtained dependence (13) is obvious [6].

In the asymptotic limit $\alpha \ll 1$ from equation (12) we get

$$\langle \rho \rangle \simeq \tfrac{1}{2}(e^{2N\alpha} - 1),$$

which was obtained earlier in [1–5].

Using equation (12) we can show that the inverse localization length of the chain consisting of N delta-like potential (2) is equal to $(N \to \infty)$

$$a\xi^{-1} = N^{-1} \ln\left(\operatorname{sh} Nx + \frac{\operatorname{ch} Nx}{\operatorname{ch} x}\right). \tag{14}$$

As is seen from equation (14) in the weak scattering limit $\alpha \ll 1$ we get

$$a\xi^{-1} \simeq 2\alpha - \frac{2\alpha^2}{N}. \tag{15}$$

The first term of the right-hand side of this equation is twice the size of the inverse localisation length whch was obtained in [14]. This means that in the weak disordered limit the well-known relationship between the various localization lengths, namely

$$\ln\langle \rho \rangle = 2\langle \ln \rho \rangle \tag{16}$$

takes place [15]. This relationship reflects the fact that the resistance is not self-averaging in a 1D disordered system.

In the other limiting case of strong scattering $\alpha \gg 1$ we get an asymptotic expression for ξ^{-1} from equation (14).

$$a\xi^{-1} \simeq \ln 4\alpha - \frac{1}{N}\left(\ln 2 - \frac{1}{2\alpha}\right). \tag{17}$$

Note that the first term of equation (14) was found in [9].

As shown in [9] the expression for $\langle \rho \rangle$, for uniformly distributed amplitudes V_n, is given by

$$\langle \ln \rho \rangle \simeq N \ln \frac{12\alpha}{l^2}. \tag{18}$$

By comparison with equations (17) and (18) one obtains that in the strong disordered limit for the relationship between the various localization lengths we get

$$\ln\langle \rho \rangle = \beta \langle \ln \rho \rangle,$$

where $1 < \beta < 2$.

We note that the same behaviour is observed for the variance

$$\sigma^2 = \langle (\ln \rho - \langle \ln \rho \rangle)^2 \rangle$$

of the logarithm of the resistance too. The σ^2 under weak-scattering conditions is twice the value of its mean [16], but the ratio in the region of strong scattering decreases with increasing disorder.

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