Chapter 4
Integers and Number Theory

Suggestions: Although it is possible to draw pictures for the models used in the following activities, encourage students to use two different colored squares to represent protons and electrons and to actually manipulate the squares when completing the activities.

Activity 1

Answers will vary; three possibilities for each integer are given below.

+5  -1  -2  0

1. a. +5  b. -1  c. -2  d. 0

2. a.  5  b.  1  c.  2  d.  0

3. a.  15  b.  12  c.  0

4. a. the number.  b. zero.  c. the opposite of the number.

Activity 2

Box 1
1. Answers may vary. Pair the HEADS with the TAILS and count the number of coins that cannot be paired. If the remaining coins are HEADS, move that many units to the right. If they are TAILS, move that many units to the left.
2. positive  3. negative
4. No. Since there are 6 pennies, the difference between the number of HEADS and the number of TAILS must always be an even number. Thus, you always move an even number of units.
5. If you toss 3 HEADS and 3 TAILS, you will end your turn at the same place you started.
6. Answers will vary; three possibilities for each integer are given below.
   a.  5H 1T  b.  7T 4H  c.  2H 2T
   4H 0T  3T 0H  3H 3T
   8H 4T  6T 3H  8H 8T

(Continued on next page.)
Activity 2 Continued

Box 2
1. a. Each pair means to move 1 unit to the right and then back 1 unit to the left, so the net result is no change in position.
   b. 2 units to the left c. -2 d. +2 + -4 = -2
2. Sketches will vary.
   a. -4 b. +2 c. 0 d. -7
3. a. negative b. The sum is the negative of the sum of the absolute values of the numbers.
4. a. The sum is 0 when the numbers are opposites.
   b. The sum is positive when the number with the greater absolute value is positive.
   c. The sum is negative when the number with the greater absolute value is negative.
5. Find the absolute values of the numbers and subtract the smaller absolute value from the larger one. The sum is the negative of this difference if the number with the greater absolute value is negative; otherwise, the sum equals this difference.
6. a. +8 b. +6 c. -55 d. 0

Activity 3

Box 1
4 + 1 = 5 1. a. If the absolute value of the positive number is greater than the absolute value of the negative number.
4 + 0 = 4  b. If the absolute value of the two addends is the same.
4 + -1 = 3  c. If the absolute value of the negative number is greater than the absolute value of the positive number.
4 + -2 = 2  d. Find the absolute values of the addends and subtract the smaller absolute value from the larger one. The sum is the negative of this difference if the number with the larger absolute value is negative; otherwise, the sum equals the difference.
4 + -3 = 1
4 + -4 = 0
4 + -5 = -1

Box 2
-4 + 0 = -4 The sum of two negative numbers is the negative of the sum of their absolute values.
-4 + -1 = -5
-4 + -2 = -6
-4 + -3 = -7
-4 + -4 = -8

EXTENSIONS
Answers will vary. Two possibilities for each situation are given below.
1. The price of a stock increased 7 points. The following day it fell 4 points. What was the net change in the stock’s price? ( +7 + -4)
   The temperature was -8°. By noon, it had risen 3°. What was the temperature at noon? (-8° + +3°)
2. The price of an ounce of gold fell $2 one day and $3 the next. What was the net change in the price of an ounce of gold?
   A football team lost 7 yards on one play and 8 on the next. How many yards did they lose on the two plays? (-7 + -8)
Activity 4

**Suggestions:** Introduce this activity by reviewing the take away interpretation of whole number subtraction and explaining how the interpretation can be extended to the integers. It is important to emphasize to students that the renaming used in the subtraction model for integers is equivalent to adding 0 to the number. It is also analogous to the renaming used in whole number subtraction, where to subtract the units digits, for example, it is often necessary to rename by “borrowing” a 10.

1. By adding 5 protons and 5 electrons to a set containing 2 electrons.

2. a. Rename "Take Away" -4
   b. Rename "Take Away" +7
   c. Rename "Take Away" -7
   d. Rename "Take Away" -1
   e. Rename "Take Away" -8
   f. Rename "Take Away" +4

3. a. less than  b. greater than
4. a. -4  b. +7  c. -7  d. -1  e. -8  f. +4
5. The answers are the same. If the problem in Exercise 2 Parts (a - f) is \( x - y \), then the corresponding problem in Exercise 4 Parts (a - f) is \( x + (-y) \).
6. To subtract an integer, add its opposite. That is, \( x - y = x + (-y) \).
7. a. +8  b. +20  c. -17  d. -22

Activity 5

1. \( 4 - 0 = 4 \)  \( 4 + 0 \)
   \( 4 - 1 = 3 \)  \( 4 + 1 \)
   \( 4 - 2 = 2 \)  \( 4 + 2 \)
   \( 4 - 3 = 1 \)  \( 4 + 3 \)
   \( 4 - 4 = 0 \)  \( 4 + 4 \)
   \( 4 - 5 = -1 \)  \( 4 + 5 \)
   \( 4 - 6 = -2 \)  \( 4 + 6 \)
   \( 4 - 7 = -3 \)  \( 4 + 7 \)

(Continued on next page.)
Activity 5 Continued

3. 4 – 3 = 1 4 + -3 4. -4 – 3 = -7 -4 + -3
4 – 2 = 2 4 + -2 -4 – 2 = -6 -4 + -2
4 – 1 = 3 4 + -1 -4 – 1 = -5 -4 + -1
4 – 0 = 4 4 + 0 -4 – 0 = -4 -4 + 0
4 – -1 = 5 4 + 1 -4 – -1 = -3 -4 + 1
4 – -2 = 6 4 + 2 -4 – -2 = -2 -4 + 2
4 – -3 = 7 4 + 3 -4 – -3 = -1 -4 + 3
4 – -4 = 8 4 + 4 -4 – -4 = 0 -4 + 4

5. See the second column in the answers for Exercises 1–4.

6. For any two integers \(a\) and \(b\), \(a – b = a + (-b)\).

EXTENSIONS

Answers will vary. Two possibilities for each situation are given below.

1. The average July temperature in Barrow, AK, is 39°. In February, the average temperature is -20°. How much warmer is the average temperature in July than in February? (+39° – -20°)
   The elevation of Mount McKinley is 20,320 ft and the elevation of Death Valley is -282 ft. What is the difference in elevation between Mount McKinley and Death Valley?
   (+20,320 – -282)

2. The average April temperature in Barrow, AK, is -2°. In February, the average temperature is -20°. How much warmer is the average temperature in April than in February? (-2° – -20°)
   A football team lost 12 yards on one play and 8 on the next. How many more yards did they lose on first play than the second? (-8 – -12)

Activity 6

1. 5; 1 – -4 = 5
2. 5 3. -2 4. -6 5. 2 6. -4 7. 8
8. -8 9. 9 10. 2 11. 6 12. -3 13. 9
14. a. negative
   b. The sum is the negative of the sum of the absolute values of the numbers.
15. a. The sum is positive when the number with the greater absolute value is positive.
   b. The sum is negative when the number with the greater absolute value is negative.
   c. The sum is 0 when the numbers are opposites.

16. Find the absolute values of the numbers and subtract the smaller absolute value from the larger one. The sum is the negative of this difference if the number with the greater absolute value is negative; otherwise, the sum equals this difference.
17. a. 3 + -7
   b. The first numbers are the same, 3, but the second numbers are opposites, 7 and -7.
18. 6 + 2 -5 + -3 3 + 6 -2 + 4
   In each case, the original problem was \(x – y\) and the related addition problem was \(x + (-y)\).
   That is, the first numbers are the same, but the second numbers are opposites.
19. -6 + - (-4) = -6 + 4
Activity 7

Box 1
1. $4 \times 3 = 12$  
2. $4 \times 5 = 20$  
3. $4 \times 2 = 8$  
4. $4 \times 1 = 4$  
5. $4 \times 0 = 0$  
6. $4 \times -1 = -4$  
7. $4 \times -2 = -8$  
8. $4 \times -3 = -12$  
9. $4 \times -4 = -16$

3. The product of a positive and a negative number is the negative of the product of the absolute values of the numbers.

4. $-3 \times 2 = -6$  
5. $3 \times -6 = -18$

6. The product of two negative numbers is the product of their absolute values.

Box 2
The quotient of two integers equals the quotient of their absolute values if both integers are positive or both are negative; otherwise, the quotient of the two integers equals the negative of the quotient of the absolute values.

Activity 8

<table>
<thead>
<tr>
<th>Number of Squares</th>
<th>Dimensions of the Rectangular Arrays</th>
<th>Total Number of Arrays</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1 \times 1$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$1 \times 2, 2 \times 1$</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>$1 \times 3, 3 \times 1$</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>$1 \times 4, 4 \times 1, 2 \times 2$</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>$1 \times 5, 5 \times 1$</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>$1 \times 6, 6 \times 1, 2 \times 3, 3 \times 2$</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>$1 \times 7, 7 \times 1$</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>$1 \times 8, 8 \times 1, 2 \times 4, 4 \times 2$</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>$1 \times 9, 9 \times 1, 3 \times 3$</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>$1 \times 10, 10 \times 1, 2 \times 5, 5 \times 2$</td>
<td>4</td>
</tr>
<tr>
<td>11</td>
<td>$1 \times 11, 11 \times 1$</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>$1 \times 12, 12 \times 1, 2 \times 6, 6 \times 2, 3 \times 4, 4 \times 3$</td>
<td>6</td>
</tr>
</tbody>
</table>

1. Number of Squares that Produced:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Only One Array</td>
<td>Only Two Arrays</td>
<td>More Than Two Arrays</td>
<td>An Odd Number of Arrays</td>
</tr>
<tr>
<td>1</td>
<td>2, 3, 5, 7, 11, ...</td>
<td>4, 6, 8, 9, 10, 12, ...</td>
<td>1, 4, 9, ...</td>
</tr>
</tbody>
</table>

2. a. 8  b. C  c. 1, 2, 3, 4, 6, 8, 12, 24
3. a. 1, 2, 4, 8, 16  b. 5  c. C, D

(Continued on next page.)
Activity 8 Continued

4. The number of factors is equal to the number of arrays.

5. a. The numbers in column D are square numbers; when a square array is rotated from horizontal to vertical, the dimensions remain the same.
   b. 16, 25, 36

6. a. prime numbers  
   b. composite numbers  
   c. square numbers

7. The square numbers; they all have an odd number of factors and the square numbers greater than 1 all have more than 2 factors. One has exactly 1 factor.

8. No

9. a. $2^2 \times 7$  
   b. $2 \times 3 \times 7$  
   c. $2 \times 3 \times 5^2$  
   d. $3 \times 7 \times 11$

10. a. Yes  
    b. The factorizations would all contain the same factors, but the order in which the factors are written could be different.

Activity 9

4 is the first multiple of 2 to fall through the sieve.
9 is the first multiple of 3 to fall through.

(Continued on next page.)
Activity 9 Continued

Box 1
1. They are prime numbers because they have exactly 2 whole number factors. When coloring the sieve, the next number that does not have any corners colored is prime, and the diamond surrounding the number is not colored.
2. The prime numbers are colored orange and have exactly one corner colored.
3. Squares that have one or more corners colored and that do not have the diamond surrounding the number colored contain composite numbers.

Box 2
1. a. 25  b. 49  2. 121  3. It is the square of the number.  4. 17  
5. 997; 999 and 998 are composite numbers since they are divisible by 3 and 2 respectively. However, 997 is not divisible by any prime number less than 1000 (2, 3, 5, 7, 9, 11, 13, 17, 19, 23, 29, or 31) so it is a prime number.

Box 3
1. 6, 12, 18, 24, 30, 36, 42, 48, 54, 66, 72, 78, 84, 90, 96  2. the multiples of 6  
3. a. Numbers colored with the codes for 2 and 7 are multiples of 14.  b. Numbers colored with the codes for 2, 3, and 5 are multiples of 30.

Box 4
1. a. $54 = 2 \times 3^3$  b. $84 = 2^2 \times 3 \times 7$  c. $100 = 2^2 \times 5^2$

Box 5
1. 3 and 5, 5 and 7, 11 and 13, 17 and 19, 29 and 31, 41 and 43, 59 and 61, 71 and 73  2. 90, 91, 92, 93, 94, 95, 96  3. $2 \times 3 \times 5 \times 7 = 210$

Activity 10

Box 1
1. The numbers on the game board are composite. The ones in the factor list are prime.
2. a. 2, 3, and 5 are the prime factors of 60.  
   4, 6, 10, 12, 15, 20, 30, and 60 are the composite factors.
   1 is only factor that is neither prime nor composite. 
   b. Place two paper clips on 2, one on 3, and one on 5. 
   c. There is only one way to place the paper clips to cover 60.

(Continued on next page.)
Activity 10 Continued

Box 2
1. In Step 1, two factors of 45 are selected to start the factor tree. The factor 5 is circled because it is a prime. The 9 is not circled because it is composite.
2. In Step 2, each composite factor from Step 1 is written as the product of two factors. The branches stop at the circled numbers because they are prime and cannot be factored further.
3. \(45 = 3 \times 3 \times 5 \) or \(3^2 \times 5\)

Box 3
1. \[\begin{align*} 120 &= 2 \times 2 \times 2 \times 3 \times 5 \text{ or } 2^3 \times 3 \times 5 \\
\end{align*}\]
2. Answers will vary. One example is given below.
3. \[\begin{align*} 120 &= 8 \times 15 \\
\end{align*}\]
4. The circled numbers, the prime factors of 120, are the same.

Box 4
1. The procedures are the same except that at each step the calculator chooses the smallest prime factor of any remaining composite number as one of the factors instead of choosing any two factors.
2. a. \(72 = 2^3 \times 3^2\)
   b. \(102 = 2 \times 3 \times 17\)
   c. \(210 = 2 \times 3 \times 5 \times 7\)
   d. \(924 = 2^2 \times 3 \times 7 \times 11\)
3. For any whole number \(N\), enter the fraction \(\frac{N}{N}\) and press the \(\text{SIMP}\) and \(\text{=}\) keys.
   If the display shows 1, the number \(N\) is prime.

Activity 11

Box 1

<table>
<thead>
<tr>
<th>Number</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factorization</td>
<td>(2^1)</td>
<td>(3^1)</td>
<td>(2^2)</td>
<td>(3^3)</td>
<td>(3^2)</td>
</tr>
<tr>
<td>Number of Factors</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

The number of factors is one more than the exponent.
Answers will vary. Examples: \(2^6 = 64\) has 7 factors. \(3^{10} = 59049\) has 11 factors.
In general, if \(p\) is a prime, then \(p^6\) has 7 factors and \(p^{10}\) has 11 factors.

(Continued on next page.)
### Activity 11 Continued

#### Box 2

<table>
<thead>
<tr>
<th>Number</th>
<th>6</th>
<th>24</th>
<th>60</th>
<th>72</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factorization</td>
<td>$2^1 \times 3^1$</td>
<td>$2^3 \times 3^1$</td>
<td>$2^2 \times 3 \times 5$</td>
<td>$2^3 \times 3^2$</td>
<td>$2^2 \times 5^2$</td>
</tr>
<tr>
<td>Number of Factors</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>12</td>
<td>9</td>
</tr>
</tbody>
</table>

To find the number of factors, add 1 to each of the exponents in the prime factorization, and find the product of the sums. [Example: $20 = 2^2 \times 5^1$; number of factors = $(2 + 1) \times (1 + 1) = 6$]

$360 = 2^3 \times 3^2 \times 5^1$  
Number of Factors = $(3 + 1) \times (2 + 1) \times (1 + 1) = 4 \times 3 \times 2 = 24$

Factors = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360

To find the total number of factors of a number, add 1 to each of the exponents in the prime factorization of the number, and find the product of the sums.

Answers will vary. Example: $240 = 2^4 \times 3 \times 5$ has 20 factors, 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 40, 48, 60, 80, 120, 240.

### Activity 12

#### Box 1

121 is a palindrome, that is, its value is unchanged when its digits are reversed.

1. Answers will vary; 1, 4, 9, and 16 are possibilities.
2. Answers will vary; 292, 8338, and 757 are possibilities.

#### Box 2

64 is also a perfect square.

1. Answers will vary; 1, 8, and 27 are possibilities.
2. 343

#### Box 3

When the digits of 13 are reversed, the number formed, 31, is also a prime number.

1. 5 (101, 131, 151, 181, 191)
2. Answers will vary; 17, 37, and 79 are possibilities.

#### Box 4

1. Answers will vary.
2. | Number | Rating | Number | Rating | Number | Rating | Number | Rating |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>576</td>
<td>35</td>
<td>756</td>
<td>31</td>
<td>864</td>
<td>31</td>
<td>936</td>
<td>31</td>
</tr>
<tr>
<td>672</td>
<td>31</td>
<td>780</td>
<td>31</td>
<td>900</td>
<td>37</td>
<td>960</td>
<td>35</td>
</tr>
<tr>
<td>720</td>
<td>33</td>
<td>792</td>
<td>31</td>
<td>924</td>
<td>31</td>
<td>990</td>
<td>31</td>
</tr>
<tr>
<td>729</td>
<td>30</td>
<td>840</td>
<td>35</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. 23 if the sum of the digits is greater than 14. (15 pts - prime, 2 pts - 2 factors, 2 pts - odd, 4 pts - sum of digits greater than 14)
4. a. 10 pts for being a perfect cube, 7 pts for being a perfect square, 2 pts for being odd (the number could also be even and score 4 pts), and 1 pt for each of the following factors:
   1, n, $\sqrt[n]{n}$, $\sqrt[3]{n}$, and $(\sqrt[3]{n})^2$.
   (There may also be other factors which would make the interest rating even higher.)
   b. $729 = 9 \times 9 \times 9$ or $27 \times 27$
   c. 30
5. 900 Interest Rating = 37
Activity 13

1. A $4 \times 8$ rectangle can be tiled with $1 \times 1$, $2 \times 2$, and $4 \times 4$ squares as shown below. The rectangle cannot be tiled with $3 \times 3$ squares since the length of a side of one square is less than the 4 unit width of the rectangle and two squares side-by-side have a length of 6 which is greater than the 4 unit width.

2. No other squares can be used to tile a $4 \times 8$ rectangle since the length of a side of the square would be greater than the 4 unit width of the rectangle.

<table>
<thead>
<tr>
<th>Dimensions of Rectangle width × length</th>
<th>Prime Factorization</th>
<th>Common Factors</th>
<th>Dimensions of Squares That May Be Used</th>
<th>Length of Side of Largest Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 × 8</td>
<td>2 × 2</td>
<td>2 × 2</td>
<td>1 × 1, 2 × 2, 4 × 4</td>
<td>4</td>
</tr>
<tr>
<td>6 × 9</td>
<td>2 × 3</td>
<td>3 × 3</td>
<td>1 × 1, 3 × 3</td>
<td>3</td>
</tr>
<tr>
<td>18 × 30</td>
<td>2 × 3 × 3</td>
<td>2 × 3 × 5</td>
<td>1 × 1, 2 × 2, 3 × 3, 6 × 6</td>
<td>6</td>
</tr>
<tr>
<td>24 × 36</td>
<td>2 × 2 × 2 × 3</td>
<td>2 × 2 × 3</td>
<td>1 × 1, 2 × 2, 3 × 3, 4 × 4, 6 × 6, 12 × 12</td>
<td>12</td>
</tr>
<tr>
<td>8 × 15</td>
<td>2 × 2 × 2</td>
<td>3 × 5</td>
<td>1 × 1</td>
<td>1</td>
</tr>
</tbody>
</table>

4. The length of a side of the largest square that can be used to tile a rectangle equals the product of the common factors.

Box 1

1. a. 1, 2, 3, 6, 9, 18  b. 1, 2, 3, 5, 6, 10, 15, 30  c. 1, 2, 3, 6  d. 6
2. a. 1, 2, 3, 4, 6, 8, 12, 24  b. 1, 2, 3, 4, 6, 9, 12, 18, 36  c. 1, 2, 3, 4, 6, 12  d. 12
3. The GCD is the same as the length of a side of the largest square that can be used to tile the rectangle.
4. The length of the side of the square is $\text{GCD}(m,n)$ which is the product of the common prime factors of $m$ and $n$.

Box 2

1. a. $\text{GCD}(105,126) = 21$  b. $\text{GCD}(38,95) = 19$
   c. $\text{GCD}(420,714) = 42$  d. $\text{GCD}(924,770) = 154$
2. Like the procedure in Exercise 4, the calculator simplifies the fraction by finding the common prime factors of the two numbers and dividing both the numerator and denominator by them.
### Activity 14

<table>
<thead>
<tr>
<th>Height</th>
<th>Base</th>
<th>No. of Hits</th>
<th>No. of Squares</th>
<th>Final Pocket</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>6</td>
<td>5</td>
<td>12</td>
<td>D</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>12</td>
<td>35</td>
<td>C</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>B</td>
</tr>
</tbody>
</table>

Number of Hits = (Base + Height) ÷ GCD(Base, Height)
Number of Squares = LCM(Base, Height)

**EXTENSION**

- J = Height ÷ GCD(Base, Height)
- K = Base ÷ GCD(Base, Height)

If J is odd and K is even, Final Pocket is B.
If J is odd and K is odd, Final Pocket is C.
If J is even and K is odd, Final Pocket is D.
The ball never goes in A.