DUALITY OF TOTALLY BOUNDED ABELIAN GROUPS

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Abstract. Let G be a totally bounded Abelian (Hausdorff) group and denote by $\widetilde{G}$ the group of *characters, i.e.,* continuous homomorphisms from G into the usual Torus $\mathbb{T}$, equipped with operation defined pointwise, and endowed with the topology of *pointwise convergence* on G. Define the *evaluation mapping* $\varphi : G \longrightarrow \widetilde{\widetilde{G}}$ by the relation $\varphi(x)(\lambda) := \lambda(x)$ for $x \in G$, $\lambda \in \widetilde{G}$. We show that φ is a topological isomorphism of G onto $\widetilde{\widetilde{G}}$. We compare this with the usual duality on locally compact Abelian (LCA) groups. As an application, a new proof is presented of the fact that LCA groups respect compactness when equipped with their Bohr topology.

1. Introduction and Motivation

All groups considered in this note are commutative. If G is a topological group, denote by $\widehat{G}$ the group of characters of G (as defined in the abstract) equipped with the topology of uniform convergence on the compact subsets of G. So defined $\widehat{G}$ becomes a topological group. Let the evaluation map $\Omega : G \to \widehat{\widehat{G}}$ be defined in a similar way as in the abstract. The *Pontryagin-van Kampen (P-vK) Duality* Theorem states that if G is locally compact and Hausdorff, then Ω is a topological isomorphism of G onto $\widehat{\widehat{G}}$. Notice that in this case, even though $\widehat{G}$

Date: August 1999, submitted for publication.

1991 *Mathematics Subject Classification.* 03E50, 22A05, 22A10, 22B05, 54D30, 54D65, 54H11.

Key words and phrases. Ascoli's Theorem, Baire space, Bohr topology, character, compact-open topology, equicontinuity, finite-open topology, pointwise convergence, Pontryagin-van Kampen duality, totally bounded group.

This paper is based on the second chapter of the first author's doctoral dissertation, and *dually,* on the first section of the second author's doctoral dissertation, both written at Wesleyan University, Middletown, Conn., under the supervision of Professor W. W. Comfort.

We thank the *Instituto de Matemáticas, Unidad Morelia,* of the *Universidad Nacional Autónoma de México*, UNAM, for generous hospitality during the Summer of 1999.

is also locally compact (and Hausdorff), G and $\widehat{G}$ can be topologically quite different, as $\widehat{G}$ is compact whenever G is discrete and *vice versa.*

The class of totally bounded Hausdorff Abelian groups, *i.e.,* of subgroups of compact Hausdorff groups, is quite stable as it is closed under forming subgroups, products and Hausdorff quotients. Moreover, their topological structure is very easy to describe: It is a theorem of COMFORT and ROSS [3] that the topology of such a group is the weakest that makes the elements of the character group continuous. This in particular implies that the character group A of G is *point-separating, i.e.,* if $g \in G$ is not the identity, then there is $a \in A$ such that $a(g) \neq 1$. The converse is evident: If G is an Abelian group and A is any *point-separating* group of homomorphisms from G into $\mathbb{T}$, then, since $\mathbb{T}$ is compact, the weakest topology on G that makes the elements of A continuous makes of it a totally bounded Hausdorff group.

Therefore an equivalent formulation of the theorem of COMFORT and ROSS is the following: *If G is a topological group, then it carries the topology of pointwise convergence on its character group if and only if G is totally bounded.* Thus while it is natural to equip the dual of a locally compact Abelian group with the *compact-open* topology, the reformulation of the COMFORT and ROSS' theorem naturally takes us to consider the *finite-open* topology on the character group of any totally bounded group.

The aim of this paper is to show that a phenomenon similar to the Pontryagin-van Kampen duality for locally compact groups applies to totally bounded groups. The precise result is given in the abstract. Then we compare these two dualities in a number of ways. As an application, in the last section of the paper we reconsider a classic theorem of GLICKSBERG to the effect that locally compact groups, when equipped with their Bohr topologies, do not create new compact sets. Many proofs of this result are available. For example, the one provided in [23] depends on the particular cases G discrete and $G = \mathbb{R}$. Theorem 12, a hybrid of our results and ASCOLI'S Theorem, is used to give new proofs of Glicksberg's theorem in these two cases.

2. NOTATION AND SETTING

We make the blank assumption that all topologies considered in this note are completely regular and Hausdorff. If $\varepsilon > 0$, and α is in $\mathbb{T}$ or $\mathbb{R}$, then $V_\varepsilon(\alpha)$ denotes the neighborhood around α of radius ε. For any group X let $\mathcal{H}(X)$, or simply $\mathcal{H}$ if it is clear what group X we are talking about, be the group of all homomorphisms from X to the circle group $\mathbb{T}$. We denote by $\mathcal{S}(\mathcal{H})$ the set of all point-separating subgroups $A \in \mathcal{H}$. If $A \in \mathcal{S}(\mathcal{H})$, we denote by $\langle X, \tau_A \rangle$ the group X endowed with the weakest topology τ_A that makes all the elements of A continuous. As mentioned in the introduction, $\langle X, \tau_A \rangle$ is a totally bounded topological group, and vice versa, if $G := \langle X, t \rangle$ is a totally bounded group with underlying group X, topology t, and character group A, then $A \in \mathcal{S}(\mathcal{H}(X))$ and $t = \tau_A$.

We denote the dual group of a topological group G by G^*, and by $\widetilde{G}$ the group G^* equipped with the topology inherited from $\mathbb{T}^X$, where X is the underlying group of G. We can look at it as follows: We define

$$e : \mathcal{H} \longrightarrow \mathbb{T}^X \quad \text{by} \;\; e(\lambda)_g := \lambda(g) \;\; (\lambda \in \mathcal{H}, g \in X),$$

and we set

$$\widetilde{G} := e[G^*] \subseteq e[\mathcal{H}] \subseteq \mathbb{T}^X.$$

So defined, $\widetilde{G}$ is a totally bounded topological group. The basic open neighborhoods of the identity in $\widetilde{G}$ have the form

$$(F, \varepsilon) := \{\lambda \in \widetilde{G} : |\lambda(g) - 1| < \varepsilon, \quad g \in F\}$$

where $\varepsilon > 0$ and $F \subseteq G$ is finite.

For example, if $H = \langle A, t \rangle$ is a topological group with underlying group A and topology t, denote by t^+ the weakest topology on A that makes every element of H^* continuous, and define $H^+ := \langle A, t^+ \rangle$. Then, setting $X := H^*$ and $G := \langle X, \tau_A \rangle$, we obtain $H^+ = \widetilde{G}$ from the above construction.

Returning to our original notation when G is a topological group, G^+ is always a continuous image of G. Most of the times however, G and G^+ are not topologically isomorphic and often G^+ is not even Hausdorff. By the COMFORT and ROSS' theorem quoted above G and G^+ are topologically isomorphic if and only if G is totally bounded. The condition on G^+ to be Hausdorff is precisely the definition of G being *maximally almost periodic* (MAP). The non-trivial fact that locally compact Abelian groups are MAP is a decisive step in the proof of the P-vK duality theorem. Examples of non MAP groups are plentiful ([8] (23.32) and [17]).

Since G^+ is a totally bounded group, it is a dense subgroup of its so-called (compact) *Weil completion*, denoted by $\overline{G^+}$. The latter is also referred to as the *Bohr compactification* of G, and it is denoted by bG, *i.e.*, $\overline{G^+}$ and bG coincide. We refer to the topology of G^+ as the *Bohr topology* of G.

3. Main Theorem

Theorem 1. *Let G be a totally bounded topological group. Define $\varphi : G \longrightarrow \widetilde{\widetilde{G}}$ by*

$$\varphi(g)(\lambda) := \lambda(g) \text{ for } g \in G, \lambda \in \widetilde{G}.$$

Then φ is a topological isomorphism of G onto $\widetilde{\widetilde{G}}$.

Proof: Clearly φ is one-to-one since $\widetilde{G}$ separates points. To see that φ is onto we follow directly the proof of Theorem 1.3 of Comfort and Ross [3]. Pick $f \in \widetilde{\widetilde{G}}$. We proceed to show the existence of $g_0 \in G$ such that $f = \varphi(g_0)$: Define a homomorphism $\tau : \widetilde{G} \longrightarrow \mathbb{T}^{\varphi[G]}$ by:

$$\tau(\lambda)_{\varphi(g)} := \varphi(g)(\lambda) = \lambda(g).$$

Clearly τ is one-to-one. Define $\theta : \tau[\widetilde{G}] \longrightarrow \mathbb{T}$ by:

$$\theta(\tau(\lambda)) := f(\lambda).$$

Then θ is a homomorphism and we claim that in fact it is continuous. For, notice that, since f itself is continuous on $\widetilde{G}$, if $\varepsilon > 0$ is given, then there is a finite subset $F := \{g_i\}_{i=1}^n$ of G, and $\eta > 0$ such that, if $\lambda \in U := (F, \eta)$, then

$$|f(\lambda) - 1| < \epsilon.$$

Let V be the basic open neighborhood of 1 in $\mathbb{T}^{\varphi[G]}$ defined as the intersection of the inverse images of the η-neighborhood of 1 in $\mathbb{T}$ under the n projections $\varphi(g_i)$. Then $V_0 := V \cap \tau[\widetilde{G}]$ is a neighborhood of 1 in $\tau[\widetilde{G}]$ and if $\tau(\lambda) \in V_0$, then

$$|\theta(\tau(\lambda)) - 1| = |f(\lambda) - 1| < \epsilon,$$

as required.

By uniform continuity θ can be extended to a character on the closure K of $\tau[\widetilde{G}]$ in $\mathbb{T}^{\varphi[G]}$. This character can in turn be extended to a character on $\mathbb{T}^{\varphi[G]}$ (see Hewitt and Ross [8], (24.12)). Let us again denote this extension as θ. By loc. cit. (23.21) there are elements $\{g_i\}_{i=1}^n$ and continuous characters of the circle $\{\theta_i\}_{i=1}^n$ such that

$$\theta((t_{\varphi(g)})) = \prod_{k=1}^{n} \theta_k(t_{\varphi(g_k)}) \text{ whenever } (t_{\varphi(g)}) \in \mathbb{T}^{\varphi[G]}.$$

Since for each character θ_k there is some $m_k \in \mathbb{Z}$ such that $\theta_k(t) = t^{m_k}$, we have for every $\lambda \in \widetilde{G}$:

$$\begin{aligned} f(\lambda) &= \theta(\tau(\lambda)) = \theta((\lambda(g))) \\ &= \prod_{k=1}^{n} \theta_k(\lambda(g_k)) = \prod_{k=1}^{n} (\lambda(g_k))^{m_k} \\ &= \lambda(g_1^{m_1} \cdots g_n^{m_n}). \end{aligned}$$

Thus define $g_0 := g_1^{m_1} \cdots g_n^{m_n}$. Then $f = \varphi(g_0)$, which proves that φ is onto.

That φ is a homeomorphism follows from the fact that the topologies of G and $\widetilde{\widetilde{G}}$ are both the topologies of pointwise convergence on $\widetilde{G}$. Hence the proof of the theorem is complete. ■

Remark 2. Theorem 1 follows as well from Propositions 2.8, 3.9 and Theorem 3.11 of Menini and Orsatti [14].

4. Examples

We start by comparing $\widehat{\widehat{G}}$ with $\widetilde{\widetilde{G}}$ in two somewhat extreme cases. In the first example $\widehat{\widehat{G}}$ is the same as $\widetilde{\widetilde{G}}$. In the second we explain how $\widehat{\widehat{G}}$ is different from $\widetilde{\widetilde{G}}$.

Example 3. If G is compact, then $\widehat{G}$ is discrete, while $\widetilde{G}$ is a dense subgroup of its (compact) completion. If $\widetilde{G}$ were discrete, it would be a locally compact subgroup of a compact group, so it would be closed and therefore finite. Hence while $\widehat{G}$ and $\widetilde{G}$ are algebraically the same, they are different as topological groups. But $\widehat{\widehat{G}} = \widetilde{\widetilde{G}}$ as both topologies come from the finite sets of $\widehat{G}$ or correspondingly $\widetilde{G}$. ■

Example 4. Let A denote any infinite discrete group and set $G := A^+$. Then G is totally bounded, hence Theorem 1 applies. We will see *infra* in Lemma 14 that all the compact subsets of G are finite (Leptin [12], see also [4] and [7]). Therefore $\widetilde{G}$ and $\widehat{G}$ are equal (to the compact group $\widehat{A}$). $\widetilde{\widetilde{G}}$ and $\widehat{\widehat{G}}$ are algebraically the same but their topologies differ: $\widetilde{\widetilde{G}}$ gets its topology from the finite subsets of $\widehat{A}$, hence it is topologically isomorphic with A^+ as Theorem 1 assures. In contrast, $\widehat{\widehat{G}}$ receives its topology from the compact subsets of $\widehat{A}$, *i.e.,* $\widehat{\widehat{G}}$ is the discrete group A. ■

Our aim next is, given an abstract group, to characterize a totally bounded topology on it in terms of the dual. We need a lemma first.

Lemma 5. *Let G_1 and G_2 be totally bounded groups. Let*

$$\psi : G_1 \longrightarrow G_2$$

be a continuous homomorphism. Then the adjoint homomorphism

$$\widetilde{\psi} : \widetilde{G_2} \longrightarrow \widetilde{G_1}$$

defined by

$$\widetilde{\psi}(\lambda) := \lambda \circ \psi$$

is also continuous.

Proof: For $F \overset{\text{finite}}{\subseteq} G_1$, we have $\psi(F) \overset{\text{finite}}{\subseteq} G_2$, hence, if $\varepsilon > 0$, then

$$\widetilde{\psi}((\psi(F), \varepsilon)) \subseteq (F, \varepsilon).$$

■

Assume X is an Abelian group; let $A, B \in S(\mathcal{H})$ be equipped each with the topology inherited from $\mathbb{T}^X$, and define $G := \langle X, \tau_A \rangle$ and $H := \langle X, \tau_B \rangle$. Notice that $\widetilde{G}$ is topologically isomorphic to A and $\widetilde{H}$ is topologically isomorphic to B.

Corollary 6. *G is topologically isomorphic to H if and only if $\widetilde{G}$ is topologically isomorphic to $\widetilde{H}$.*

Proof: ($\Longrightarrow$) Lemma 5.
($\Longleftarrow$) Loc. cit. and Theorem 1. ■

Corollary 7. *Two totally bounded groups G and H are topologically isomorphic if and only if $\widetilde{G}$ and $\widetilde{H}$ are topologically isomorphic.* ■

Example 8. Let G and H be two infinite cyclic totally bounded groups. Since the abstract groups of G, H and $\mathbb{Z}$ are the same, and since the only two automorphisms of $\mathbb{Z}$ are those sending $1 \to \pm 1$, we see that G and H are topologically isomorphic if and only if $G = H$. In particular any two dense cyclic subgroups of $\mathbb{T}$, say A and B, are topologically isomorphic if and only if $A = B$. In contrast, notice that if A and B are countable subgroups of $\mathbb{T}$ then, by the celebrated result of Sierpiński [21], all four groups A, B, $\langle \mathbb{Z}, \tau_A \rangle$ and $\langle \mathbb{Z}, \tau_B \rangle$ are always homeomorphic. ■

We can also characterize the Bohr topology of a MAP group in terms of the second dual.

Corollary 9. *If G is MAP, then G^+ is topologically isomorphic to $\widetilde{\widetilde{G}}$.* ■

Remark 10. Recall that if G is a topological group, and $S \subseteq G$, then the *annihilator of S in $\widetilde{G}$* is defined by

$$\mathbb{A}(\widetilde{G}, S) := \{\lambda \in \widetilde{G} : \lambda[S] = \{1\}\}.$$

We have that annihilators are always *closed subgroups of $\widetilde{G}$.* In the following assume that G is a totally bounded group with H a closed subgroup of G.

(1) $(G/H)^{\widetilde{}}$ is topologically isomorphic to $\mathbb{A}(\widetilde{G}, H)$ by the map $\psi \mapsto \psi \circ \pi$ where $\pi : G \longrightarrow G/H$ stands for the natural epimorphism; compare with [8] (23.25).
(2) H is topologically isomorphic to $\mathbb{A}(\widetilde{\widetilde{G}}, \mathbb{A}(\widetilde{G}, H)) = \mathbb{A}(G, \mathbb{A}(\widetilde{G}, H))$; compare with [loc. cit.] (24.10).
(3) $\widetilde{H}$ is topologically isomorphic to $\widetilde{G}/\mathbb{A}(\widetilde{G}, H)$; compare with [loc. cit.] (24.11).

We left the proofs of the above assertions to the reader.

5. An application

In this section we apply the material so far developed to reconsider the following fact, first noticed in [7]:

Theorem 11. (Glicksberg) *If G is a locally compact Abelian group, then G and G^+ have the same compact sets.*

A MAP group that satisfies the conclusion of Glicksberg's theorem is said to *respect compactness*. Additional proofs of this result were obtained by Christensen [5], Dikranjan, Prodanov, and Stoyanov [6] (3.4.3), Hughes [9], Moran [13], and Trigos-Arrieta [23]. The special cases G discrete (proven for the first time by Leptin [12], and reproved by Comfort and Trigos-Arrieta in [4]), and $G = \mathbb{R}$ are of fundamental importance in the proof offered in [23]. The case G discrete is treated in cases in [4]. In what follows we deal with it all at once. The case $G = \mathbb{R}$ relies upon [22] (1.7), a result that remains unpublished. A modification of it (Lemma 15 *infra*) is offered here. We believe however that what makes our approach interesting is the following:

Theorem 12. *A topological group G satisfying P-vK duality respects compactness if and only if the compact subsets of G^+ coincide with the subsets of $\widetilde{\widetilde{G}}$ $(= G)$ whose restrictions to the compact subsets of $\widehat{G}$ are equicontinuous,* i.e., *if Bohr compactness and equicontinuity on the compact subsets of $\widehat{G}$ coincide.*

Recall that $K \subseteq \widetilde{\widetilde{G}}$ is *equicontinuous* on the subset C of $\widehat{G}$ if whenever $\varepsilon > 0$ is given, then for every $c \in C$, $\bigcap_{\lambda \in K} \lambda_{|C}^{\leftarrow}[V_\varepsilon(\lambda(c))]$ is a neighborhood of c in C.

Proof of Theorem 12. By Corollary 9 G^+ is topologically isomorphic to $\widetilde{\widetilde{G}}$. Also G is topologically isomorphic to $\widehat{\widehat{G}}$ by P-vK duality. Thus, noticing that $\widetilde{\widetilde{G}} = \widetilde{\widehat{G}}$, it is clear that G respects compactness if and only if every compact subset K of $\widetilde{\widehat{G}}$, is also compact on $\widehat{\widehat{G}}$. Since $\mathbb{T}$ is compact, Ascoli's Theorem (see for example [2] (Theorem X §2.5 2)), assures that this is equivalent to show that K is equicontinuous on every compact subset C of $\widehat{G}$. ■

We start with the following fact.

Lemma 13. *Let C be a compact group. If $\langle \lambda_n \rangle_{n \in \mathbb{N}}$ converges to* 1 *in $\widetilde{C}$, then $K_0 := \{\lambda_n : n \in \mathbb{N}\} \cup \{1\}$ is equicontinuous on C.*

Proof: If $\varepsilon > 0$, and $\alpha \in \mathbb{T}$, let $B_\varepsilon(\alpha) := \{\zeta \in \mathbb{T} : |\zeta - \alpha| \leq \varepsilon\}$. Set $C_n := \bigcap_{k \geq n} \lambda_k^{\leftarrow}[B_{\varepsilon/2}(1)]$. Since $\lambda_n \longrightarrow 1$ pointwise on C, we have $C = \bigcup_{n \in \mathbb{N}} C_n$. Since C is Baire, there exists $N_0 \in \mathbb{N}$ and an open set, say U, with $U \subseteq C_{N_0}$. Since C is compact, there exist finitely many elements, say $x_1, ..., x_m$, in C such that $C = x_1 U \cup ... \cup x_m U$. Now choose $N \geq N_0$ such that $n \geq N \implies |\lambda_n(x_j) - 1| < \varepsilon/2$, for every $j = 1, ..., m$.

Let $c \in C$. Choose $j \in \{1, ..., m\}$ such that $c \in x_j U$. Choose $u \in U$ with $c = x_j u$. If $n \geq N$, then

$$\begin{aligned} |\lambda_n(c) - 1| &= |\lambda_n(x_j u) - 1| = |\lambda_n(x_j)\lambda_n(u) - 1| \\ &\leq |\lambda_n(x_j)\lambda_n(u) - \lambda_n(u)| + |\lambda_n(u) - 1| \\ &= |\lambda_n(x_j) - 1| + |\lambda_n(u) - 1| < \varepsilon. \end{aligned}$$

Since $W := \bigcap_{n<N} \lambda_n^{\leftarrow}[V_\varepsilon(\lambda_n(c))]$ is a neighborhood of c in C, it follows that $V := W \cap x_j U$ is a neighborhood of c as well, totally contained in $\bigcap_{\lambda \in K_0} \lambda^{\leftarrow}[V_\varepsilon(\lambda(c))]$, as required. ■

The above result remains true assuming only that C is Baire and taking continuous homomorphisms into some topological group H rather than characters, as proved by Khan and Khan [10]. Our proof however is not very long and we include it for the self-containement of the paper.

Recall that *every countable compact space is metric.* To see this, for each pair of distinct points in the space find a real-valued continuous function that separates them. Since the collection of functions found in this manner is countable, we can then consider the evaluation map defined by this collection.

Corollary 14. *If G is discrete and K is a compact subset of G^+, then K is finite.*

Proof: We argue by *reductio ad absurdum.* If K is infinite, choose an infinite countable subgroup H of G such that $H \cap K$ is infinite. By [8] (23.26), H^+ is closed in G^+. Hence $H^+ \cap K$ is a countable compact subspace of H^+. Thus there is an infinite convergent sequence $\langle \lambda_n \rangle_{n\in\mathbb{N}}$ in $H^+ \cap K$, and by a simple translation, we can even assume that $\langle \lambda_n \rangle_{n\in\mathbb{N}}$ converges to 1 in H^+, hence in G^+. By Lemma 13 we see that $K_0 := \{\lambda_n : n \in \mathbb{N}\} \cup \{1\}$ is equicontinuous on $C := \widehat{G}$. Thus, ASCOLI'S Theorem (*i.e.,* Theorem 12) implies that K_0 is a compact subspace of $\widehat{\widehat{G}}$ which is discrete, contradicting the fact that K_0 is infinite. ■

Instead of using Lemma 13 and Theorem 12 in the proof offered in [4], one uses the fact that *if G is discrete, then any convergent sequence in G^+ must be eventually constant.* This clearly follows from Lemma 13 and Theorem 12. As we pointed it out, the proof of this fact in [4] is given in cases.

For $G = \mathbb{R}$, we need the following construction in which $((x))$ stands for the decimal part of (real) x. Our model for $\mathbb{T}$ is $[0, 1)$ equipped with operation $+$ mod 1:

Lemma 15. *Let $\langle x_n \rangle_{n\in\mathbb{N}}$ be a sequence of real numbers such that $x_1 \geq 1$, and $x_{n+1} \geq (n+1)x_n$ for every $n \in \mathbb{N}$. Assume that $\langle u_n \rangle_{n\in\mathbb{N}}$ is a sequence in $[0, 1)$. Define two sequences of real numbers $\langle A_n \rangle_{n\in\mathbb{N}}$ and $\langle a_n \rangle_{n\in\mathbb{N}}$ in the following way: Put $A_1 := 0$, and $a_1 := u_1 - ((A_1)) = u_1$, and if A_k and a_k have already been defined, put*

$$A_{k+1} := \frac{x_{k+1}}{x_k}(A_k + a_k)$$

and $a_{k+1} := u_{k+1} - ((A_{k+1}))$. Then

$$\alpha := \sum_{k=1}^{\infty} \frac{a_k}{x_k}$$

is a well defined real number such that for every $k \in \mathbb{N}$

$$\left| \alpha - \sum_{j=1}^{k} \frac{a_j}{x_j} \right| \leq \frac{1}{kx_k}, \tag{A}$$

and

$$|((\alpha x_n)) - u_n| \longrightarrow 0. \tag{B}$$

Proof: Notice that, as defined, for each $k \in \mathbb{N}$ we have $|a_k| < 1$ and $A_k + a_k = [A_k] + u_k$, where $[x]$ denotes the greatest integer contained in (real) x. We must verify that $|((\alpha x_n)) - u_n| \longrightarrow 0$. For fixed $n \in \mathbb{N}$

one proves by induction on k that $x_{n+k} \geq \frac{(n+k)!}{n!} x_n$, for every $k \in \mathbb{N}$. Moreover, it is known that $\sum_{k=n+1}^{\infty} \frac{1}{k!} \leq \frac{1}{n \cdot n!}$ ([11] (7.2)). Thus

$$\begin{aligned}
\left| \alpha - \sum_{j=1}^{k} \frac{a_j}{x_j} \right| &= \left| \sum_{j=k+1}^{\infty} \frac{a_j}{x_j} \right| < \sum_{j=k+1}^{\infty} \frac{1}{x_j} \\
&= \sum_{j=1}^{\infty} \frac{1}{x_{(k+j)}} \leq \sum_{j=1}^{\infty} \frac{k!}{(k+j)! x_k} \\
&= \frac{k!}{x_k} \sum_{j=1}^{\infty} \frac{1}{(k+j)!} = \frac{k!}{x_k} \sum_{j=k+1}^{\infty} \frac{1}{j!} \\
&\leq \frac{k!}{x_k} \frac{1}{k \cdot k!} = \frac{1}{k x_k},
\end{aligned}$$

which proves (A), hence $|x_k \alpha - \sum_{j=1}^{k} a_j \frac{x_k}{x_j}| \longrightarrow 0$. We now claim that

$$\sum_{j=1}^{k} a_j \frac{x_k}{x_j} = A_k + a_k \tag{1}$$

for each $k \in \mathbb{N}$. The statement is obvious for $k = 1$. Assume that (1) holds for some $k \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{j=1}^{k+1} a_j \frac{x_{k+1}}{x_j} &= x_{k+1} \left(\sum_{j=1}^{k} \frac{a_j}{x_j} + \frac{a_{k+1}}{x_{k+1}} \right) \\
&= \frac{x_{k+1}}{x_k} \left(\sum_{j=1}^{k} a_j \frac{x_k}{x_j} \right) + a_{k+1} \\
&= \frac{x_{k+1}}{x_k} (A_k + a_k) + a_{k+1} \\
&= A_{k+1} + a_{k+1},
\end{aligned}$$

which proves the claim. It already was noticed that $A_k + a_k = [A_k] + u_k$, thus (B) follows. ■

This result is proved in [22] (1.7 & 1.8) and its proof is based on Exercise 5.37 of Parent [15].

Corollary 16. *If K is a compact subset of $\mathbb{R}^+$, then K is compact in $\mathbb{R}$.*

Proof: It is enough to show that K is bounded. If otherwise, then according to Theorem 12, we must find a compact subspace C of $\mathbb{R} = \widehat{\mathbb{R}}$, such that the restriction of K to C is not equicontinuous. Assume without loss of generality that K does not have an upper bound. Choose a sequence of real numbers $\langle x_n \rangle_{n\in\mathbb{N}}$ contained in K such that $x_1 \geq 1$, and $x_{n+1} \geq (n+1)x_n$ for every $n \in \mathbb{N}$. Consider the sequence

$$u_n := \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

The number $\alpha := \sum_{k=1}^{\infty} \frac{a_k}{x_k} \in \mathbb{R} = \widehat{\mathbb{R}}$, as supplied by Lemma 15, satisfies $|\alpha(x_n) - u_n| \longrightarrow 0$. Set $C := [\alpha - 1, \alpha + 1]$. If the restriction of K to C were equicontinuous, there would exist $\delta \in (0,1)$ such that for every $n \in \mathbb{N}$ and every $y \in V_\delta(\alpha)$, we would have $|x_n(y) - x_n(\alpha)| < \frac{1}{4}$.

However this is not the case: Otherwise, find $N \in \mathbb{N}$ such that $\frac{2}{Nx_N} < \delta$. Define the sequence

$$v_n := \begin{cases} u_n & \text{if } n \leq N \\ u_{n+1} & \text{otherwise.} \end{cases}$$

Now apply Lemma 15 again to the sequences $\langle x_n \rangle_{n\in\mathbb{N}}$, and $\langle v_n \rangle_{n\in\mathbb{N}}$, to find $b_k \in (-1,1)$ such that $\beta := \sum_{k=1}^{\infty} \frac{b_k}{x_k}$ satisfies $|\beta(x_n) - v_n| \longrightarrow 0$. Notice that $a_k = b_k$ for $k \leq N$. Therefore

$$\begin{aligned} |\alpha - \beta| &= \left| \sum_{j=N+1}^{\infty} \frac{a_j - b_j}{x_j} \right| \\ &\leq \sum_{j=N+1}^{\infty} \frac{2}{x_j} \\ &= 2 \sum_{j=N+1}^{\infty} \frac{1}{x_j} \\ &\leq 2 \frac{1}{Nx_N} < \delta. \end{aligned}$$

by Lemma 15 (A). Thus $\beta \in V_\delta(\alpha)$. Now find $M > N$ such that $|\alpha(x_n) - u_n| < \frac{1}{8}$ and $|\beta(x_n) - v_n| < \frac{1}{8}$ whenever $n \geq M$. But since $n > N \implies |u_n - v_n| = \frac{1}{2}$, $|x_n(\beta) - x_n(\alpha)| < \frac{1}{4}$ is impossible. This contradiction shows that the compactness of K in $\mathbb{R}^+$ implies the boundedness of K. ■

Remark 17. Let us denote by $\mathcal{K}$ the class of MAP groups that respect compactness. Glicksberg's Theorem says that locally compact groups are contained in $\mathcal{K}$. Remus and Trigos-Arrieta have proved that $\mathcal{K}$ is productive [18] (2.1), and hereditary with respect to subgroups [20]. Additional results are given in [1] and [19]. However a full characterization of $\mathcal{K}$ is still underway.

Acknowledgment. We are grateful to Professor W. W. Comfort, for stimulating conversations regarding this article. We also express our gratitude to Professor Dikran Dikranjan for letting us know about the information contained in Remark 2. Our thanks are due also to Professor Dieter Remus for answering a question regarding this article.

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