TOTALLY BOUNDED TOPOLOGICAL GROUP TOPOLOGIES ON THE INTEGERS

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ABSTRACT. We generalize an argument of W. W. COMFORT, F. J. TRIGOS-ARRIETA and T. S. WU [7] showing that if there is a non-trivial sequence converging to the identity in a locally compact Abelian group G, then $A := \{\lambda \in \widehat{G} : \lambda(x_n) \to 1\}$ is a locally μ-null subgroup of the character group $\widehat{G}$ of G, where μ denotes *Haar* measure on $\widehat{G}$.

Using a result of the same authors we show the existence of families $\mathcal{A}$ and $\mathcal{B}$ of dense subgroups of $\mathbb{T} \simeq \widehat{\widehat{\mathbb{Z}}}$ such that

(i) $|\mathcal{A}| = |\mathcal{B}| = 2^{\mathfrak{c}}$;

(ii) each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$ is algebraically isomorphic to the free Abelian group $\oplus_{\mathfrak{c}} \mathbb{Z}$;

(iii) the spaces $< \mathbb{Z}, \tau_A > (A \in \mathcal{A})$ are pairwise non-homeomorphic, and the spaces $< \mathbb{Z}, \tau_B > (B \in \mathcal{B})$ are pairwise non-homeomorphic (by τ_X we denote the weakest topology making all elements of X continuous);

(iv) each group $< \mathbb{Z}, \tau_A > (A \in \mathcal{A})$ has a non-trivial convergent sequence; and

(v) every convergent sequence of $< \mathbb{Z}, \tau_B > (B \in \mathcal{B})$ is trivial.

1. INTRODUCTION AND MOTIVATION

Given an abelian group G and a non-trivial sequence u_n in G, when is it possible to construct a Hausdorff group topology on G that allows this sequence to converge? This question, especially for the integers,

has been studied by many authors GRAEV [11] (§8, theorem 6), NIEN-
HUYS [22], ZELENYUK and PROTASOV [28], CLARK and CATES [3] and
others. It is known that the answer is positive for any sequence (in the
integers or the rationals) that satisfies appropriate "sparseness" con-
ditions (e.g., while the sequence of squares in the integers $\mathbb{Z}$ cannot
converge to 0 in any Hausdorff group topology [3], any sequence u_n in
$\mathbb{Z}$ with $\lim |u_{n+1}/u_n| = \infty$ allows such a topology (TRIGOS-ARRIETA
[27], also see RACZKOWSKI and TRIGOS-ARRIETA [25]), a complete de-
scription in the general case as well as many interesting examples can
be found in [28]. The purpose of this paper is to provide some insights
into this question in the realm of *totally bounded* group topologies.

All groups that we consider here are Abelian and all topologies are
Hausdorff.

We consider a locally compact Abelian group G and a countably infi-
nite subset S with no accumulation points. We show that the subgroups
of continuous characters of G that induce on G a (necessarily totally
bounded) group topology in which S forms a sequence converging to
the identity, must be *locally μ-null* with respect to any *Haar* measure
μ on $\widehat{G}$ (Definition 5). Although for discrete groups this means that
such (sub)groups of characters are of measure 0, we give an example of
when such a subgroup is of infinite measure (any locally μ-null subset
is either of measure 0 or of infinite measure, cf HEWITT and ROSS [12]
11.33).

An argument in BERHANU, COMFORT and REID [2], (Corollary 5.7),
shows that on any Abelian group of cardinality α, one can induce
2^{2^α}-many distinct totally bounded, topological group topologies. This
means, for example, that on the group of integers $\mathbb{Z}$ there are $2^{\mathfrak{c}}$-many
distinct such topologies. Of these, $\mathfrak{c}$-many are metric (i.e., of weight ω
cf [2] Theorem 4.3). SIERPIŃSKI [26] has shown that countable, dense
in itself, metrizable spaces are all homeomorphic to each other. Thus
all metric topologies on $\mathbb{Z}$ are all homeomorphic to each other as spaces.

On the other hand we have, on $\mathbb{Z}$, $2^{\mathfrak{c}}$-many distinct totally bounded
group topologies of weight $\mathfrak{c}$. On any countable set one can have only
$\mathfrak{c}$-many distinct permutations. Since a homeomorphism is among other
things a permutation, this implies that we have on $\mathbb{Z}$, $2^{\mathfrak{c}}$-many pairwise

non-homeomorphic, totally bounded group topologies of weight $\mathfrak{c}$. The goal of this paper is to study these kinds of topologies.

Notation: We will adopt the notation from COMFORT [5]. By $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ we denote the groups of integers, rationals and reals, respectively. In each case the operation is addition and the topology is the usual, unless otherwise specified.

For an Abelian group G we denote by G_d the group G equipped with the discrete topology. By $wt(G)$ we denote the weight of the topological group G, i.e., the minimal cardinality of a base for the topology on G.

The brackets $< \cdots >$ will serve a double purpose, since confusion is unlikely. By $< G, \tau >$ we denote the group G with topology τ, while for a subset $S \subseteq G$, $< S >$ denotes the group generated by S. For a real number x, let $[x]$ denote the integer part of x, and $\{x\}$ the fractional part of x. We identify $\mathbb{T}$ with the multiplicative subgroup $\{e^{2\pi i x} : 0 \leq x < 1\}$ of the complex plane. We denote by $\mathcal{H}(G) := Hom(G, \mathbb{T})$, the group of all homomorphisms of G into $\mathbb{T}$.

Definition 1. A *character* of a topological group G is a continuous homomorphism $h : G \to \mathbb{T}$.

For a topological group G, we denote by $\widehat{G}$ the group of all characters. We equip $\widehat{G}$ with the compact-open topology, i.e., the topology whose basic open neighborhoods of the identity in $\widehat{G}$ are of the form:

$$U(F, \epsilon) := \{\lambda \in \widehat{G} : |\lambda(g) - 1| < \epsilon, \quad g \in F\}$$

where $F \overset{\text{cpt}}{\subseteq} G$ and $\epsilon > 0$ is arbitrary.

It is proved in PONTRYAGIN [23] and VAN KAMPEN [17] that the dual of a locally compact Abelian group (from now on **LCA**) is itself **LCA**. Moreover G is discrete if and only if $\widehat{G}$ is compact. In particular, the dual group of $\mathbb{Z}$ is $\mathbb{T}$ (see [12] (23.27 b.)).

Definition 2. A subgroup A of $\mathcal{H}(G)$ is said to separate points of G if for every g in $G \setminus \{0\}$ there is h in A such that $h(g) \neq 1$.

It is a theorem of COMFORT and ROSS [6] that every totally bounded topological group topology is the weakest topology on G that makes all elements of a suitably chosen point-separating (equivalently, dense in $\widehat{G_d}$) subgroup A of $\mathcal{H}(G)$ continuous; and conversely, every such topology is totally bounded. We denote it as $< G, \tau_A >$. In particular

we have a one-to-one correspondence between dense subgroups of $\mathbb{T}$ and totally bounded topological group topologies of $\mathbb{Z}$. Our objective is to study what properties of dense subgroups A of $\mathbb{T}$ influence what topological properties of $< \mathbb{Z}, \tau_A >$.

For example, if G is a countable Abelian group, by the above-mentioned Theorem of SIERPIŃSKI [26] we know that it is sufficient for point separating subgroups A and B to satisfy $|A| = |B| = \aleph_0$, for the spaces $< G, \tau_A >$ and $< G, \tau_B >$ to be homeomorphic. By [5], Sect. 4.2, the corresponding conclusion can fail when, for example, the point-separating subgroups are of cardinality $\mathfrak{c}$.

Now consider the usual *Lebesgue measure* on $\mathbb{T}$. In this paper we are able to make a very crude distinction, namely (a) we have constructed a Lebesgue measurable dense subgroup $A \subseteq \mathbb{T}$ of cardinality $\mathfrak{c}$ for which $< \mathbb{Z}, \tau_A >$ has a non-trivial sequence converging to 0, hence has infinite compact sets; (b) we have shown that an algebraically isomorphic copy B of such subgroup A can be positioned in $\mathbb{T}$ in such a way as to be no longer Lebesgue measurable, hence by [7], inducing a totally bounded topological group topology, τ_B whose convergent sequences are all trivial. Thus we have an example of non-homeomorphic spaces $< \mathbb{Z}, \tau_A >$ and $< \mathbb{Z}, \tau_B >$, such that A and B are isomorphic as groups. Since a compact countable space is second countable (see HODEL [16] 7.2) for countable groups to have all compact sets finite is equivalent to having no non-trivial converging sequences. The above fails for groups of higher cardinality, as there are examples of groups that have compact sets of cardinality $\mathfrak{c}$, without non-trivial converging sequences (see EDA, OHTA and YAMADA [9], Theorem 3.4).

Additionally, we show that on $\mathbb{Z}$, there are $2^{\mathfrak{c}}$-many pairwise *non-homeomorphic* totally bounded group topologies in each of which the only convergent sequences are the trivial ones. We also show that on $\mathbb{Z}$ there are $2^{\mathfrak{c}}$-many pairwise non-homeomorphic totally bounded group topologies all sharing the same convergent sequence.

and simplified at least one proof, acknowledged *in situ.* For this I thank the referee.

2. Bohr topologies and convergent sequences

Using analytic apparatus we show in this section that any subgroup A of characters of any **LCA** group G that makes some fixed, but arbitrary sequence with no accumulation points in G, convergent in $< G, \tau_A >$, is locally null with respect to Haar measure on $\widehat{G}$. This generalizes in particular the second conclusion in Lemma 3.10 of [7].

For $G \in$ **LCA**, denote by μ its *normalized* Haar measure (see [12] §15). Let $\mathcal{L}_1(G)$ be the collection of μ-integrable continuous complex-valued functions on G. For any topological space X, let $\mathcal{C}_0(X)$ denote the set of all complex-valued continuous functions f on X such that for every $\epsilon > 0$, there exists a compact set F of X, depending on f and ϵ, such that $|f(x)| < \epsilon$ for all $x \in X \setminus F$.

Definition 3. For $f \in \mathcal{L}_1(G)$ define $\widehat{f} : \widehat{G} \to \mathbb{C}$ by

$$\widehat{f}(\lambda) := \int_G f(x)\lambda(x)\, d\mu.$$

The function $\widehat{f}$ is called the *Fourier transform* of f.

We denote by $A(\widehat{G})$ the space of all *Fourier transforms* of functions in $\mathcal{L}_1(G)$. The standard topology of $\widehat{G}$ is the compact-open topology. An alternative way to describe it is as the *weak topology* determined by $A(\widehat{G})$, that is the weakest topology for which all the functions in $A(\widehat{G})$ are continuous; see LOOMIS [21] pg. 151, or KATZNELSON [18] pg. 190. Such a topology is referred to as the *Gel'fand topology* (cf [12] C.23). Using the above, we are now ready to state the generalized *Riemann-Lebesgue Lemma* (see FOLLAND [10] Proposition 4.13 or HEWITT and ROSS [13] Remark 28.42.):

Lemma 4. *Let $G \in$ **LCA**. Then for all $f \in \mathcal{L}_1(G)$, we have $\widehat{f} \in \mathcal{C}_0(\widehat{G})$.*

Following [12], Definition 11.26, we say:

Definition 5. Let X be a topological space and m a measure defined on it. If $A \subseteq X$ and $m(A) = 0$ we say that A is a *m-null* set. If $A \cap F$ is *m-null* for every $F \overset{\text{cpt}}{\subseteq} X$, then A is said to be *locally m-null.*

Any locally m-null set is measurable (see [12] Theorem 11.30). The proof of our next Theorem generalizes the construction of KUIPERS and NIEDERREITER [19], Theorem 7.8.

Theorem 6. *Let $G \in \mathbf{LCA}$ and let ρ denote a Haar measure on $\widehat{G}$. Let $\{x_n\}_{n<\omega}$ be a faithfully indexed sequence in G with no accumulation points and define*

$$A := \{\lambda \in \widehat{G} : \lambda(x_n) \longrightarrow 1\}.$$

Then A is a locally ρ-null subgroup of $\widehat{G}$.

Proof: We need to show that $\rho(A \cap F) = 0$ for all $F \overset{\mathrm{cpt}}{\subseteq} \widehat{G}$. Notice that the elements of G can be seen as functions on $\widehat{G}$. If $\lambda \in A$, then

$$\lim_{n \longrightarrow \infty} x_n(\lambda) = 1$$

and $|x_n(\lambda)| \leq 1_{|F}(\lambda)$ for all $n < \omega$ and all $\lambda \in F$. By *Lebesgue*'s Theorem on Dominated Convergence, we have the equality

$$\lim_{n \longrightarrow \infty} \int_{A \cap F} x_n(\lambda)\, d\rho = \int_{A \cap F} \lim_{n \longrightarrow \infty} x_n(\lambda)\, d\rho = \rho(A \cap F),$$

as it is easy to see (i.e., first conclusion of Lemma 3.10 in [7]) that A is measurable. Clearly then $1_{A \cap F} \in \mathcal{L}_1(\widehat{G})$, hence from Lemma 4 and *Pontryagin Duality* we have $\widehat{1}_{A \cap F} \in \mathcal{C}_0(G)$, and since $(x_n)_{n<\omega}$ has no accumulation points

$$\lim_{n \longrightarrow \infty} \int_{A \cap F} x_n(\lambda)\, d\rho = \lim_{n \longrightarrow \infty} \int_{\widehat{G}} 1_{A \cap F}(\lambda) x_n(\lambda)\, d\rho = \lim_{n \longrightarrow \infty} \widehat{1}_{A \cap F}(x_n) = 0.$$

In other words $\rho(A \cap F) = 0$. $\qquad\square$

The above proposition applies only to **LCA** groups that are non-compact, since in a compact space every infinite set has at least one accumulation point. As a corollary, we get the conclusion of Lemma 3.10 of [7], referring to the measure of A.

Corollary 7. *Let G be a* discrete *Abelian group and let ρ denote a Haar measure on $\widehat{G}$. Let $\{x_n\}_{n<\omega}$ be a faithfully indexed sequence in G, and define*

$$A := \{\lambda \in \widehat{G} : \lambda(x_n) \longrightarrow 1\}.$$

Then A is a ρ-measurable subgroup of $\widehat{G}$ and $\rho(A) = 0$.

Moreover, if G is a **LCA** group, then $\widehat{G}$ is σ-compact if and only if G is metric (see Hewitt and Stromberg [15], Theorem 3, or [5], 3.14). Thus we have:

Corollary 8. *If G is **LCA** and metric and A is defined as above, then $\rho(A) = 0$.* $\square$

Below we will show that there is a **LCA** group G with dense subgroups of $\widehat{G}$ of infinite measure that induce on G totally bounded topological group topologies with non-trivial convergent sequences. Recall that the group of real numbers $\mathbb{R}$ is algebraically isomorphic with $\bigoplus_{\alpha < \mathfrak{c}} \mathbb{Q}$. $\widehat{\mathbb{Q}_d}$ can be realized as $\mathbf{\Sigma_a}$ where $\mathbf{a} = (2, 3, \dots)$ (see [12] (25.4)), thus by *loc. cit.* (23.21) we have

$$\widehat{\mathbb{R}_d} \simeq (\mathbf{\Sigma_a})^{\mathfrak{c}}.$$

Example 9. Let $B \stackrel{\text{dense}}{\subseteq} \mathbb{R} \simeq \widehat{\mathbb{R}}$ be a group that induces on $\mathbb{R}$ a topology in which the sequence $(n!)_{n<\omega}$ converges to 0 (see [24] Theorem 15). Define $G := (\mathbf{\Sigma_a})^{\mathfrak{c}} \times \mathbb{R}$. Then G is a **LCA** group in which the sequence $((0, n!))_{n<\omega}$ has no accumulation points. We have $\widehat{G} \simeq \mathbb{R}_d \times \mathbb{R}$. Let $A := \mathbb{R}_d \times B$. Then A separates the points of G, since $\mathbb{R}_d$ separates the points of $(\mathbf{\Sigma_a})^{\mathfrak{c}}$ and B separates the points of $\mathbb{R}$. Also in $< G, \tau_A >$ the sequence $((0, n!))_{n<\omega}$ converges to $(0, 0)$. Hence by Theorem 6, A is ρ-measurable. It is in fact a locally ρ-null subgroup of $\widehat{G}$. But the measure of A is infinite since the measure of each set of the form $\mathbb{R}_d \times \{b\}$ is infinite in $\widehat{G}$ (see [12] 11.33).

3. Totally bounded group topologies on $\mathbb{Z}$ with non-trivial convergent sequences

We know that whenever $A \stackrel{\text{dense}}{\subseteq} \mathbb{T}$ with $|A| = \aleph_0$, then $< \mathbb{Z}, \tau_A >$ is a metric group. As such, it necessarily contains convergent sequences. On the other hand, in $< \mathbb{Z}, \tau_{\mathbb{T}} >$, often referred to as $\mathbb{Z}^{\#}$, every convergent sequence is trivial (see Leptin [20]). In [7] it is shown that $\mathbb{Z}$ may admit a totally bounded topological group topology τ, in which every convergent sequence is trivial, yet this topology is not the topology inherited from the Bohr compactification of $\mathbb{Z}_d$. Note that the character group A of $< \mathbb{Z}, \tau >$ is (dense and) uncountable.

Thus it is natural to ask whether there exists a subgroup $A \overset{\text{dense}}{\subseteq} \mathbb{T}$ with $|A| > \aleph_0$ such that $< \mathbb{Z}, \tau_A >$ has non-trivial convergent sequences. The positive answer follows from:

Theorem 10. *Let $(u_n)_{n \in \mathbb{N}}$ be any sequence in $\mathbb{Z}$ such that for all $n \in \mathbb{N}$*

$$u_{n+1}/u_n \geq n + 1.$$

Then there exists a totally bounded, topological group topology on $\mathbb{Z}$ of weight $\mathfrak{c}$ in which $u_n \to 0$.

Proof: With no loss of generality assume that $u_n > 0$ for all $n \in \mathbb{N}$. We will construct a dense subgroup A of $\mathbb{T}$ of cardinality $\mathfrak{c}$ that induces on $\mathbb{Z}$ a (necessarily totally bounded) topological group topology τ_A in which $u_n \to 0$.

Let $C := \{\xi \in \{0,1\}^{\mathbb{N}} : \xi_1 = 0\}$. For each $\xi \in C$ let us define by induction a sequence of intervals $(I_n(\xi))_{n \in \mathbb{N}}$ in $\mathbb{R}$ as follows:

(1) $I_1(\xi) := [0, 2)$

(2) Suppose the interval $I_n(\xi) := [p_n(\xi), p_n(\xi) + 2/n)$ has been defined.

(3) Let $q_n(\xi)$ be the unique integer lying in the interval

$$[p_n(\xi)u_{n+1}/u_n, p_n(\xi)u_{n+1}/u_n + 1)).$$

Set $I_{n+1}(\xi) := [q_n(\xi) + \xi_n, q_n(\xi) + \xi_n + 2/(n+1))$.

Now define the compact intervals J_n (obtained by shrinking the closures of the intervals I_n)

$$J_n(\xi) := [p_n(\xi)/u_n, (p_n(\xi) + 2/n)/u_n].$$

Then we have $J_{n+1}(\xi) \subseteq J_n(\xi)$ because:

$$(**) \qquad \frac{p_n(\xi)}{u_n} \leq \frac{q_n(\xi) + \xi_n}{u_{n+1}} \leq \frac{q_n(\xi) + \xi_n + \frac{2}{n+1}}{u_{n+1}} \leq \frac{p_n(\xi) + \frac{2}{n}}{u_n}.$$

The middle of the inequality is obvious. To see the first one, notice that by the choice of $q_n(\xi)$ we have

$$\frac{p_n(\xi)u_{n+1}}{u_n} \leq q_n(\xi).$$

Then it follows that

$$\frac{p_n(\xi)}{u_n} \leq \frac{q_n(\xi)}{u_{n+1}} \leq \frac{q_n(\xi) + \xi_n}{u_{n+1}}.$$

To see the last inequality in (∗∗), by the choice of $q_n(\xi)$ we have

$$q_n(\xi) < \frac{p_n(\xi)u_{n+1}}{u_n} + 1,$$

thus

$$\frac{q_n(\xi)}{u_{n+1}} < \frac{p_n(\xi)}{u_n} + \frac{1}{u_{n+1}},$$

hence

$$\frac{q_n(\xi) + \xi_n + \frac{2}{n+1}}{u_{n+1}} < \frac{p_n(\xi)}{u_n} + \frac{2 + \frac{2}{n+1}}{u_{n+1}}.$$

The rest follows from the choice of the sequence u_n, for since we have

$$\frac{u_{n+1}}{u_n} \geq n+1 \quad \Leftrightarrow \quad \frac{1}{u_{n+1}} \leq \frac{1}{u_n(n+1)},$$

a quick check shows that

$$\frac{2n+4}{(n+1)^2} \leq \frac{2}{n}.$$

Then the rest of the inequality follows, for

$$\frac{q_n(\xi) + \xi_n + \frac{2}{n+1}}{u_{n+1}} < \frac{p_n(\xi)}{u_n} + \frac{2n+4}{u_n(n+1)^2} \leq \frac{p_n(\xi) + \frac{2}{n}}{u_n}.$$

Thus (∗∗) holds. The nested sequence $\{J_n(\xi)\}_{n\in\mathbb{N}}$ of compact intervals has non-empty intersection. Let $y_\xi \in \bigcap_{n\in\mathbb{N}} J_n(\xi)$. We claim that the numbers y_ξ ($\xi \in C$) are distinct. To see this let $\xi \neq \epsilon$, and say that k is the first integer so that $\xi_k \neq \epsilon_k$. Then, since $I_k(\xi) \cap I_k(\epsilon) = \emptyset$, we have $J_k(\xi) \cap J_k(\epsilon) = \emptyset$, and hence $y_\xi \neq y_\epsilon$.

Since for all $n \in \mathbb{N}$ we have $y_\xi u_n \in I_n(\xi)$, and $p_n(\xi)$ is an integer, then if $n \geq N$ we have $\{y_\xi u_n\} \in [0, 2/N]$. Define $\lambda_\xi : \mathbb{Z} \to \mathbb{T}$ by $\lambda_\xi(x) = e^{2\pi i y_\xi x}$. Thus for all $\xi \in C$, $\lambda_\xi(u_n) \to 1$. Now define

$$A := < \{\lambda_\xi\}_{\xi \in C} > .$$

Then $A \overset{\text{dense}}{\subseteq} \mathbb{T}$ and by Corollary 7, every $\lambda \in A$ satisfies $\lambda(u_n) \to 1$. Since $|A| = \mathfrak{c}$, the topology τ_A is as required. □

By Theorem 6, the subgroup A of $\mathbb{T}$ constructed above is always measurable, hence by the Steinhaus-Weil Theorem (cf HEWITT and STROMBERG [14] 10.43) it is of measure 0.

Corollary 11. *There exists a totally bounded, topological group topology τ on $\mathbb{Z}$ of weight $\mathfrak{c}$ in which the sequence $(n!)_{n\in\mathbb{N}}$ converges to 0.*

$\square$

It was brought to our attention by the referee that if A is a dense subgroup of $\mathbb{T} = \widehat{\mathbb{Z}}$ in which $n! \longrightarrow 0$ in $(\mathbb{Z}, \tau_A)$, then $a^{n!} \longrightarrow 1$ for every $a \in A$. The subgroup of *all* such elements $a \in \mathbb{T}$ is denoted by $\mathbb{T}!$ in ARMACOST [1], Def. 3.1. This yields another proof of the Corollary by considering $\tau = \tau_{\mathbb{T}!}$ and noticing that $|\mathbb{T}!| = \mathfrak{c}$ (see [8], Theorem 4.4.2, p. 150, where $\mathbb{T}!$ is proved to consist precisely of those $\sum_{i=2}^{\infty} x_i/i!$, in which $\lim_{i \to \infty} x_i/i = 0$; clearly this is satisfied for $x_i = 0$ or 1).

Corollary 12. *Given any faithfully indexed sequence $(u_n)_{n \in \mathbb{N}}$ in $\mathbb{Z}$ there is a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ and a totally bounded topological group topology τ, on $\mathbb{Z}$ such that*
(a) $wt(< \mathbb{Z}, \tau >) = \mathfrak{c}$, and
(b) $(u_{n_k})_{k \in \mathbb{N}}$ converges to 0 in $< \mathbb{Z}, \tau >$.

Proof: Take any infinite sequence in $\mathbb{Z}$. With no loss of generality assume it has infinitely many positive terms, and extract an infinite subsequence $(u_n)_{n \in \mathbb{N}}$ so that for all $n \in \mathbb{N}$

$$\frac{u_{n+1}}{u_n} \geq n + 1.$$

Now apply Theorem 10. $\square$

It is interesting to understand whether the condition $u_{n+1}/u_n \geq n+1$ in Theorem 10 is really necessary. Let us see that one cannot prove the theorem for $u_n = p^n$, where p is a prime. The following is due to the referee:

Theorem 13. *Every totally bounded, topological group topology on $\mathbb{Z}$ in which the sequence $p^n \to 0$ is metrizable.*

Proof: Suppose that $\tau = \tau_A$ for some subgroup A of $\mathbb{T} = \widehat{\mathbb{Z}}$ and $p^n \longrightarrow 0$ in $(\mathbb{Z}, \tau)$. Clearly, $\chi(p^n) \longrightarrow 1$ in $\mathbb{T}$ for every $\chi \in A$, i.e., identifying A with a subgroup of $\mathbb{T}$, for every $a \in A$ one must have $a^{p^n} \longrightarrow 1$ in $\mathbb{T}$. But then a is p-torsion (Lemma 2.6 of [1]), so that $A \leq \mathbb{Z}(p^\infty)$. Since $\mathbb{Z}(p^\infty)$ has no proper infinite subgroups,we get $A = \mathbb{Z}(p^\infty)$. Hence τ is the p-adic topology. In particular, τ is metrizable. $\square$

Question 1. Suppose $A \overset{\text{dense}}{\subsetneq} \mathbb{T}$ with A measurable, i.e., of measure 0. Must it follow that $< \mathbb{Z}, \tau_A >$ has non-trivial convergent sequences?

By Corollary 7, in case $A = \mathbb{T}$ or when A is non-measurable we know that in $< \mathbb{Z}, \tau_A >$ every convergent sequence is trivial.

Lemma 14. *If A is a subgroup of $\mathbb{T}$ with $|A| = \mathfrak{c}$, then A contains an isomorphic copy of $\bigoplus\limits_{\alpha < \mathfrak{c}} \mathbb{Z}_\alpha$.*

Proof: Because $|A| = \mathfrak{c}$ and $\mathbb{T}$ contains only $\aleph_0$-many torsion elements, A contains $\mathfrak{c}$-many *independent* non-torsion elements (see [12] A13). The group generated by these elements is as required. □

Corollary 15. *There are $2^{\mathfrak{c}}$-many distinct totally bounded group topologies on $\mathbb{Z}$ in each of which there is a non-trivial sequence converging to 0.*

Proof: Consider the group A that generates the topology given in Corollary 11. It has cardinality $\mathfrak{c}$. By above Lemma, A contains a copy of the free abelian group $\bigoplus\limits_{\alpha < \mathfrak{c}} \mathbb{Z}_\alpha$, which has $2^{\mathfrak{c}}$-many distinct subgroups. Since the topology induced by each of these subgroups is weaker than the one induced by A, as described in Corollary 11, it must have a convergent sequence. □

From the above Corollary follows in fact that there are $2^{\mathfrak{c}}$- many pairwise non-homeomorphic totally bounded group topologies on $\mathbb{Z}$ in each of which some fixed sequence is convergent (witness the sequence $(n!)_{n \in \mathbb{N}}$).

4. Totally bounded group topologies on $\mathbb{Z}$ in which every convergent sequence is trivial

In this section we will show that there exist $2^{\mathfrak{c}}$-many distinct totally bounded group topologies on $\mathbb{Z}$ in each of which every convergent sequence is trivial. A slight variation of the construction of [7], Theorem 3.12, yields the following:

Lemma 16. *There exists a non-measurable subgroup M of $\mathbb{T}$ such that algebraically*

$$M = \bigoplus_{\alpha < \mathfrak{c}} \mathbb{Z}_\alpha.$$

Proof: Let $\{F_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all uncountable, closed subsets of $\mathbb{T}$. There are only $\mathfrak{c}$-many such sets since the topology of $\mathbb{T}$ can be generated by a countable collection of intervals (those with rational end-points). Since every F_ξ satisfies $|F_\xi| = \mathfrak{c}$ (see CANTOR [4]) and $\mathbb{T}$ contains only $\aleph_0$-many torsion elements, one can choose non-torsion elements $p_0, q_0 \in F_0$ with $< \{q_0\} > \cap < \{p_0\} >= \{1\}$. Recursively, if $\xi < \mathfrak{c}$ and p_η, q_η, have been chosen for all $\eta < \xi$, choose non-torsion elements $p_\xi, q_\xi \in F_\xi$ with

$$< \{p_\xi\} > \cap < \{p_\eta : \eta < \xi\} \cup \{q_\eta : \eta < \xi\} >= \{1\},$$

$$< \{q_\xi\} > \cap < \{p_\eta : \eta \leq \xi\} \cup \{q_\eta : \eta < \xi\} >= \{1\}.$$

The group $M :=< \{p_\xi : \xi < \mathfrak{c}\} >$ is isomorphic with $\bigoplus_{\alpha < \mathfrak{c}} \mathbb{Z}_\alpha$ (see [12] A.18), thus $(M_d)^{\wedge} \simeq \mathbb{T}^{\mathfrak{c}}$ and M is not measurable because: (a) $\lambda(M) > 0$ is impossible, since the inequality implies that M is open in $\mathbb{T}$ (by the Steinhaus-Weil Theorem (cf [14] 10.43)) and then $M = \mathbb{T}$; (b) $\lambda(M) = 0$ is impossible, because then $\mathbb{T} \setminus M$ is measurable and $\lambda(\mathbb{T} \setminus M) > 0$ thus implying, by regularity of λ, that there is an uncountable compact subset $F = F_\xi$ of $\mathbb{T} \setminus M$, but then $p_\xi \in F \subseteq (\mathbb{T} \setminus M) \cap M = \emptyset$. $\qquad\square$

By Corollary 7, the topology induced by a non-measurable subgroup of $\mathbb{T}$ produces only trivial convergent sequences.

Proposition 17. *There are $2^{\mathfrak{c}}$-many pairwise non-homeomorphic totally bounded group topologies on $\mathbb{Z}$ in each of which every convergent sequence is trivial.*

Proof: Consider the set

$$\{p_\xi, q_\xi : \xi < \mathfrak{c}\}$$

constructed in the proof of Lemma 16. Notice that by choice of the elements this set is independent. Define $A := \{q_\xi : \xi < \mathfrak{c}\}$. Let $\{A_\varsigma\}_{\varsigma < 2^{\mathfrak{c}}}$

be an enumeration of all proper subsets of A, and consider

$$M_\varsigma :=< \{p_\xi\}_{\xi<\mathfrak{c}} \cup A_\varsigma > .$$

Then, by the Steinhaus-Weil Theorem (cf [14] 10.43)

$$\{M_\varsigma\}_{\varsigma<2^\mathfrak{c}}$$

is a family of $2^\mathfrak{c}$-many distinct non-measurable subgroups of $\mathbb{T}$, each of the form $\bigoplus_{\alpha<\mathfrak{c}} \mathbb{Z}_\alpha$. Since $M \subseteq M_\varsigma$, each M_ς induces on $\mathbb{Z}$ a required topology. This completes our proof. $\qquad\square$

The referee has contributed the result below which strengthens a fact from a previous version of this paper, the proof simplifies the old one.

Proposition 18. *Every subgroup A of $\mathbb{T}$ with $|A| = \mathfrak{c}$ is isomorphic to a non-measurable subgroup B of $\mathbb{T}$.*

Proof: Let $A_1 = \langle S \rangle + t(A)$, where $t(A)$ is the torsion part of A and S is a *maximal* linearly independent subset of A. Let M be as defined in Lemma 16. There exists an isomorphism $i : \langle S \rangle \longrightarrow M$ that can be extended to a monomorphism $i_1 : A_1 \longrightarrow M + t(\mathbb{T})$, since the sum $\langle S \rangle + t(A)$ defining A_1 is direct and the same holds true for $M + t(\mathbb{T})$. Since $\mathbb{T}$ is divisible i_1 can be extended to a homomorphism $i_2 : A \longrightarrow \mathbb{T}$. By the choice of S every non-trivial subgroup of A non-trivially meets A_1. In particular, $\ker i_2 = \{0\}$. Thus $B = i_2(A) \cong A$ and obviously contains M. To see that B is a proper subgroup of $\mathbb{T}$ we recall a property of the subgroup M mentioned implicitly in the proof of Proposition 17. Namely, the quotient $\mathbb{T}/M$ is non-torsion (as it contains an isomorphic copy of the free group $\langle \{q_\xi : \xi < \mathfrak{c}\} \rangle$). On the other hand, the quotient $A/\langle S \rangle$ is torsion by the choice of S. Therefore, $i_2(\langle S \rangle) = M$ implies that also B/M is torsion. This proves that $B \neq \mathbb{T}$. $\qquad\square$

Remark. Isomorphic subgroups of $\mathbb{T}$ can induce non-homeomorphic group topologies on $\mathbb{Z}$. For example, let A_0 be a subgroup of the group A, as constructed in the proof of Corollary 11, of the form $\bigoplus_{\alpha<\mathfrak{c}} \mathbb{Z}_\alpha$ (Lemma 14). By Theorem 6, A_0 is measurable, as $n! \to 0$ in τ_{A_0}. On the other hand if B is a non-measurable isomorphic copy of A_0, as in Proposition 18, in τ_B there are no non-trivial convergent sequences.

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