Math 4200 HW9 Solution

#10.88 Refer to Ex. 10.2, Table 1 in Appendix III is used too.

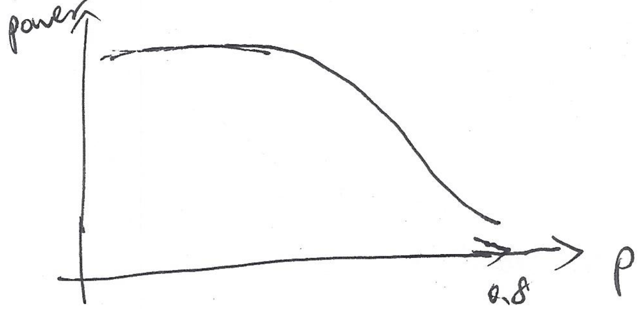
(a) power $(0.4) = P(X \le 12 \mid p = 0.4) = 0.979$

(b) power $(0.5) = P(X \le 12 \mid p = 0.5) = 0.86$

(c) power $(0.6) = P(X \le 12 \mid p = 0.6) = 0.584$

(d) power $(0.7) = P(X \le 12 \mid p = 0.7) = 0.228$

(e).



#10.91 $X_i \overset{iid}{\sim} N(\mu, 5)$. $n = 20$. Test. $H_0 : \mu = 7$ v.s. $H_1 : \mu > 7$

(a). This is similar to the class example. So detailed work is skipped here.

The R.R. given by the UMP test is $\{R.R: Z = \frac{\bar{X} - 7}{\sqrt{5/20}} > Z_{0.05}\}$

or, equivalently $\{R.R.\ \bar{X} > (1.645)\sqrt{0.25} + 7 = 7.82\}$

(b) The power function is: power $(\mu) = P(\bar{X} > 7.82 \mid \mu) = P(Z > \frac{7.82 - \mu}{\sqrt{5/20}})$

Thus,

power $(7.5) = P(\bar{X} > 7.82 \mid \mu = 7.5) = P(Z > 0.64) = 0.2611$

power $(8.0) = P(\bar{X} > 7.82 \mid \mu = 8.0) = P(Z > -0.36) = 0.6406$

power $(8.5) = P(\bar{X} > 7.82 \mid \mu = 8.5) = P(Z > -1.36) = 0.9131$

power $(9.0) = P(\bar{X} > 7.82 \mid \mu = 9.0) = P(Z > -2.36) = 0.9909$.

#10.95 $\quad f(x|\theta) = \begin{cases} \left(\frac{1}{2\theta^3}\right) x^2 e^{-x/\theta}, & x > 0 \\ 0 & \text{o.w.} \end{cases}$ $\qquad \underline{n=4}$

(a) To test $H_0: \theta = \theta_0$ v.s. $H_1: \theta = \theta_1$ where $\theta_1 > \theta_0$

$$L(\theta) = (2\theta^3)^{-n} \left(\prod_{i=1}^{n} x_i\right)^2 e^{-\sum_{i=1}^{4} x_i/\theta}$$

$$\frac{L(\theta_0)}{L(\theta_1)} = \left(\frac{\theta_1}{\theta_0}\right)^{12} e^{-\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\sum_{i=1}^{4} x_i} < k$$

This simplifies to $T = \sum_{i=1}^{4} x_i > \boxed{\ln k \left(\frac{\theta_0}{\theta_1}\right)^{12} \left[\frac{1}{\theta_0} - \frac{1}{\theta_1}\right]^{-1}}$ k^*

Note that $X_i \overset{\text{under } H_0}{\sim} \text{Gamma}(\alpha = 3, \beta = \theta_0)$

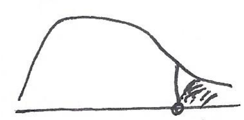
use MGF to show that $T = \sum_{i=1}^{4} X_i \sim \text{Gamma}(12, \theta_0)$

Then use transformation method / MGF to show that

$\frac{2T}{\theta_0} \sim \chi^2(24)$. (in the previous SA)

Thus, $P(\text{Type I error}) = P(T > k^* \text{ when } \theta = \theta_0) = P\left(\frac{2T}{\theta_0} > \frac{2k^*}{\theta_0}\right)$

$= P\left(\chi^2(24) > \frac{2k^*}{\theta_0}\right) = \alpha$



$\chi^2_\alpha(24)$

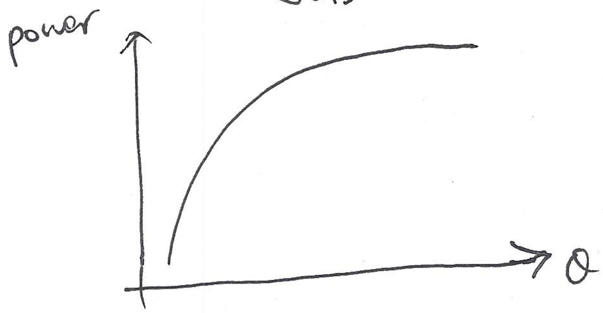
$\Rightarrow \frac{2k^*}{\theta_0} = \chi^2_\alpha(24) \quad \Rightarrow k^* = \frac{\theta_0 \cdot \chi^2_\alpha(24)}{2}$

Therefore, the R.R. given by N-P-Lemma is $\left\{RR: \sum_{i=1}^{4} x_i > \frac{\theta_0}{2} \cdot \chi^2_\alpha(24)\right\}$

(b). Since the critical region doesn't depend on any specific $\theta_1 < \theta_0$, the test is UMP.

#10.96 $f(x|\theta) = \begin{cases} \theta x^{\theta-1} & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$ where $\theta > 0$

(a) $\text{power}(\theta) = \int_{0.5}^{1} \theta x^{\theta-1} dx = 1 - 0.5^{\theta}$



(b). To test $H_0: \theta = 1$ v.s. $H_1: \theta = \theta_1$ where $\theta_1 > 1$.

The likelihood ratio is $\dfrac{L(\theta = \theta_0 = 1)}{L(\theta = \theta_1)} = \dfrac{1}{\theta_1 x^{\theta_1 - 1}} < k$

This simplifies to $x > \underbrace{\boxed{\left(\dfrac{1}{\theta_1 k}\right)^{\frac{1}{\theta_1 - 1}}}}_{k^*}$

$p(\text{Type I error}) = P(x > k^* | \theta = 1) = \int_{k^*}^{1} dx = 1 - k^* = \alpha$

$k^* = 1 - \alpha$

Therefore, the MP test given by N-P lemma has the R.R.

$\{ R.R: X > 1 - \alpha \}$.

Since the R.R. doesn't depend on a specific $\theta_1 > 1$, it is also UMP

#10.98

$$f(x|\theta) = \begin{cases} \left(\frac{1}{\theta}\right) m x^{m-1} e^{-x^m/\theta} & x>0 \\ 0 & \text{o.w.} \end{cases}$$

(a). Test $H_0: \theta = \theta_0$ v.s. $H_1: \theta = \theta_1$ where $\theta_0 < \theta_1$

$$\frac{L(\theta_0)}{L(\theta_1)} = \left(\frac{\theta_1}{\theta_0}\right)^n \exp\left[-\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\sum_{i=1}^n x_i^m\right] < K$$

This simplifies to $\sum_{i=1}^n x_i^m > \boxed{\frac{-\left[\ln K + n\ln\left(\frac{\theta_0}{\theta_1}\right)\right]}{\frac{1}{\theta_0} - \frac{1}{\theta_1}}}$ K^*

So, the R.R. has the form $\{T = \sum_{i=1}^m x_i^m > K^*\}$

Note that $\frac{2T}{\theta_0} \sim \chi^2(2n)$ (since Y^m is exponential)

Thus, $P(\text{Type I error}) = P(T > K^* \text{ when } \theta = \theta_0) = P\left(\frac{2T}{\theta_0} > \frac{2K^*}{\theta_0}\right)$

$$= P\left(\chi^2(2n) > \frac{2K^*}{\theta_0}\right) = \alpha$$

$$\Rightarrow \frac{2K^*}{\theta_0} = \chi^2_\alpha(2n) \Rightarrow K^* = \frac{\theta_0 \cdot \chi^2_\alpha(2n)}{2}$$

The MP test of $H_0: \theta = \theta_0$ v.s. $H_1: \theta = \theta_1$ (where $\theta_0 < \theta_1$) has the R.R. given by R.R $= \{\sum_{i=1}^m x_i^m > \frac{\theta_0}{2}\chi^2_\alpha(2n)\}$

Also, since the R.R. doesn't depend on specific θ_1 value, the test is also UMP.

#10.99 $X_i \overset{iid}{\sim}$ poisson (λ)

(a) $H_0: \lambda=\lambda_0$ v.s. $H_1: \lambda=\lambda_1$ where $\lambda_1>\lambda_0$.

$$\frac{L(\lambda_0)}{L(\lambda_1)}=\left(\frac{\lambda_0}{\lambda_1}\right)^{\sum_{i=1}^{n}x_i}\exp[n(\lambda_1-\lambda_0)]<K$$

This simplifies to $T=\sum_{i=1}^{n}x_i > \boxed{\dfrac{\ln k-n(\lambda_1-\lambda_0)}{\ln\left(\frac{\lambda_0}{\lambda_1}\right)}}\ k^*$.

(b). $T=\sum_{i=1}^{n}x_i \sim$ poisson $(n\lambda)$ (use MGF method).

$P(\text{Type I error})=P(T>k^* \mid \lambda=\lambda_0)=\alpha$.

(c) Since the critical value doesn't depend on the specific $\lambda_1>\lambda_0$

so the test is ump.

(d) $\{R.R: T<k^*\}$

#101 $X_i \overset{iid}{\sim} Exp(\theta)$.

Test $H_0: \theta = \theta_0$ v.s. $H_1: \theta = \theta_1$

$$\frac{L(\theta_0)}{L(\theta_1)} = \left(\frac{\theta_1}{\theta_0}\right)^n \exp\left[-\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\sum_{i=1}^n x_i\right] < K$$

this is, $\displaystyle\sum_{i=1}^n x_i < \boxed{\left[n \ln\left(\frac{\theta_0}{\theta_1}\right) + \ln K\right]\left[\frac{1}{\theta_1} - \frac{1}{\theta_0}\right]^{-1}}$ K^*

$X_i \sim Exp(\theta) \Leftrightarrow X_i \sim Gamma(\alpha=1, \beta=\theta)$

use the result from Ex 10.95. then $\sum_{i=1}^n X_i \sim Gamma(n, \theta)$

Then $\dfrac{2\sum X_i}{\theta_0} \sim \chi^2(2n)$

$$P(\text{Type I error}) = P(\sum X_i < K^* \mid \text{when } \theta = \theta_0) = P\left(\frac{2\sum X_i}{\theta_0} < \frac{2K^*}{\theta_0}\right)$$

$$= P\left(\chi^2(2n) < \frac{2K^*}{\theta_0}\right) = \alpha \qquad \Rightarrow \frac{2K^*}{\theta_0} = \chi^2_{1-\alpha}(2n)$$

$$\Rightarrow K^* = \left(\frac{\theta_0}{2}\right)\chi^2_{1-\alpha}(2n).$$

Thus, the MP test of $H_0: \theta = \theta_0$ v.s. $H_1: \theta = \theta_1$ has the R.R: $\left\{\sum_{i=1}^n X_i < \left(\frac{\theta_0}{2}\right)\chi^2_{1-\alpha}(2n)\right\}$

Since the R.R. doesn't depend on specific θ_1 value, hence it is also a UMP test.

#10.105 $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ where μ is unknown.

$H_0: \sigma^2 = \sigma_0^2$ v.s. $H_1: \sigma^2 > \sigma_0^2$.

$$\begin{cases} \Omega_0 = \{\sigma^2 : \sigma^2 = \sigma_0^2\} \\ \Omega = \{\sigma^2 : \sigma^2 \geq \sigma_0^2\} \end{cases}$$

The restricted MLE under H_0 is $\hat{\hat{\sigma}}^2 = \sigma_0^2$ and $\hat{\hat{\mu}} = \bar{X}$.

$L(\mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{\Sigma(X_i-\mu)^2}{2\sigma^2}}$ (increasing with σ^2.)

The unrestricted MLE for σ^2 is:
$$\begin{cases} \sigma_0^2 & \text{if } \sigma_0^2 > \frac{1}{n}\sum_{i=1}^n (X_i-\bar{X})^2 \\ \frac{1}{n}\sum_{i=1}^n (X_i-\bar{X})^2 & \text{if } \sigma_0^2 < \frac{1}{n}\sum_{i=1}^n (X_i-\bar{X})^2 \end{cases}$$

the unrestricted MLE for μ is: $\hat{\mu} = \bar{X}$

Thus, the likelihood ratio statistic is:

$$\lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} = \frac{L(\hat{\hat{\mu}}, \hat{\hat{\sigma}}^2)}{L(\hat{\mu}, \hat{\sigma}^2)} = \begin{cases} 1 & \text{if } \sigma_0^2 > \frac{1}{n}\sum_{i=1}^n (X_i-\bar{X})^2 \\ \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \exp\left[-\frac{\sum_{i=1}^n (X_i-\bar{X})^2}{2\sigma_0^2} + \frac{\sum_{i=1}^n (X_i-\bar{X})^2}{2\hat{\sigma}^2}\right] & \text{if } \sigma_0^2 < \frac{1}{n}\sum_{i=1}^n (X_i-\bar{X})^2 \end{cases}$$

That is.

$$\lambda = \begin{cases} 1 \\ \left[\frac{\Sigma(X_i-\bar{X})^2}{n\sigma_0^2}\right]^{n/2} \exp\left[-\frac{\sum_{i=1}^n (X_i-\bar{X})^2}{2\sigma_0^2} + \frac{n}{2}\right] & \text{if } \sigma_0^2 < \frac{\sum_{i=1}^n (X_i-\bar{X})^2}{n} \end{cases}$$

H_0 is rejected when $\lambda \leq K$. This test is a function of the chi-square test statistic $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$ and since the function is monotonically decreasing function of χ^2, the test $\lambda \leq K$ is equivalent to $\chi^2 \geq C$, where C is chosen so that the test is of size α.

#10.112 $X_i \overset{iid}{\sim} N(\mu_1, \sigma^2)$, $Y_j \overset{iid}{\sim} N(\mu_2, \sigma^2)$ where μ_1, μ_2, σ^2 are unknown.

$\boxed{\begin{array}{l}\Omega_0 = \mu_1 = \mu_2 \\ \Omega = \mu_1 - \mu_2 \geq 0\end{array}}$

Test: $H_0 : \mu_1 = \mu_2$ v.s. $H_1 : \mu_1 - \mu_2 > 0$.

under H_0, the restricted MLE are

$$\hat{\hat{\mu}} = \frac{n_1 \bar{X} + n_2 \bar{Y}}{n}, \quad \hat{\hat{\sigma}}_0^2 = \frac{1}{n}\left(\sum_{i=1}^{n_1}(X_i - \hat{\hat{\mu}})^2 + \sum_{i=1}^{n_2}(Y_i - \hat{\hat{\mu}})^2\right)$$

Next, the unrestricted MLE for the parameters are

$$\hat{\mu}_1 = \bar{X}, \quad \hat{\mu}_2 = \bar{Y}, \quad \hat{\sigma}^2 = \frac{1}{n}\left(\sum_{i=1}^{n_1}(X_i - \bar{X})^2 + \sum_{i=1}^{n_2}(Y_i - \bar{Y})^2\right)$$

The LRT test statistic

$$\lambda = \left(\frac{\hat{\sigma}^2}{\hat{\hat{\sigma}}^2}\right)^{n/2} \leq K, \text{ or equivalently reject if } \left(\frac{\hat{\hat{\sigma}}^2}{\hat{\sigma}^2}\right) \geq K^*$$

Fact: $\sum^{n_1}(X_i - \hat{\hat{\mu}})^2 = \sum^{n_1}(X_i - \bar{X} + \bar{X} - \hat{\hat{\mu}})^2 = \sum^{n_1}(X_i - \bar{X})^2 + n_1(\bar{X} - \hat{\hat{\mu}})^2$

$\sum^{n_2}(Y_i - \hat{\hat{\mu}})^2 = \sum^{n_2}(Y_i - \bar{Y} + \bar{Y} - \hat{\hat{\mu}})^2 = \sum^{n_2}(Y_i - \bar{Y})^2 + n_2(\bar{Y} - \hat{\hat{\mu}})^2$

also since $\hat{\hat{\mu}} = \frac{n_1}{n}\bar{X} + \frac{n_2}{n}\bar{Y}$, and the alternative expression

for $\hat{\hat{\sigma}}^2$ is $\sum^{n_1}(X_i - \bar{X})^2 + \sum^{n_2}(Y_i - \bar{Y})^2 + \frac{n_1 n_2}{n}(\bar{X} - \bar{Y})^2$.

$$\Rightarrow \frac{\hat{\hat{\sigma}}^2}{\hat{\sigma}^2} = 1 + \left(\frac{n_1 n_2}{n}\right)\left(\frac{(\bar{X} - \bar{Y})^2}{\sum^{n_1}(X_i - \bar{X})^2 + \sum^{n_2}(Y_i - \bar{Y})^2}\right)$$

The LRT rejects if $\frac{(\bar{X} - \bar{Y})^2}{\sqrt{\sum^{n_1}(X_i - \bar{X})^2 + \sum^{n_2}(Y_i - \bar{Y})^2}}$ is large.

This is equivalent to the two-sample t test.

#77 $X_i \overset{iid}{\sim} \text{Unif}(0, 3\theta)$.

$$\mu_1' = E(X) = \frac{3}{2}\theta$$

$$m_1' = \bar{X}$$

$$\frac{3}{2}\theta = \bar{X} \Rightarrow \hat{\theta}_{MOM} = \frac{2}{3}\bar{X}$$

#78 $\mu_1' = E(X) = \int_0^3 \alpha x^{\alpha} 3^{-\alpha} dx = \frac{3\alpha}{\alpha+1}$

$$m_1' = \bar{X}$$

$$\frac{3\alpha}{\alpha+1} = \bar{X} \Rightarrow \hat{\theta}_{MOM} = \frac{\bar{X}}{3-\bar{X}}$$

#82 (a) $L(\theta) = \theta^{-n} r^n \left(\prod_{i=1}^n x_i\right)^{r-1} \exp\left(-\sum_{i=1}^n x_i^r/\theta\right)$

$U = \sum_{i=1}^n X_i^r$ is suff stat for θ

(b) $l(\theta) = \ln(L(\theta)) = -n\ln\theta + n\ln r + (r-1)\ln\left(\prod_{i=1}^n x_i\right)$

$$- \sum_{i=1}^n x_i^r/\theta$$

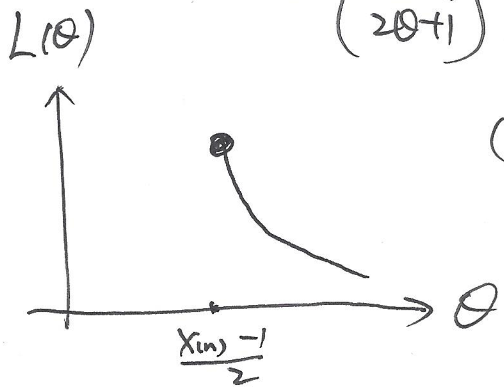
$$\frac{\partial l(\theta)}{\partial \theta} = 0 \Rightarrow \hat{\theta}_{MLE} = \frac{1}{n}\sum_{i=1}^n X_i^r$$

(c) Note that $\hat{\theta}_{MLE}$ is a function of the sufficient statistic.
Since it's easily shown that $E(X^r) = \theta$,
Then $\hat{\theta}_{MLE}$ is unbiased and the MVUE for θ.

#83 $f(x|\theta) = \frac{1}{2\theta+1} I(x \le 2\theta+1)$

(a) $\mathcal{L}(\theta) = \left(\frac{1}{2\theta+1}\right)^n I(X_{(n)} \le 2\theta+1)$

$= \left(\frac{1}{2\theta+1}\right)^n I\left(\frac{X_{(n)}-1}{2} \le \theta\right)$



(same as class example 2)

$\hat{\theta}_{MLE} = \frac{X_{(n)}-1}{2}$ maximize $L(\theta)$.

(b) $V(X) = \frac{(2\theta+1)^2}{12}$,

by the invariance property of MLE

the MLE for $V(X) = \frac{(2\theta+1)^2}{12}$ is $\frac{(X_{(n)})^2}{12}$.

#88 $L(\theta) = (\theta+1)^n \left(\prod_{i=1}^n x_i\right)^\theta$

$\ell(\theta) = n\ln(\theta+1) + \theta\left(\ln\left(\prod_{i=1}^n x_i\right)\right)$

$= n\ln(\theta+1) + \theta\left(\sum_{i=1}^n \ln x_i\right)$

$\frac{\partial \ell(\theta)}{\partial\theta} = \frac{n}{\theta+1} + \sum_{i=1}^n \ln x_i = 0$

$\Rightarrow \hat{\theta}_{MLE} = \frac{-n}{\sum_{i=1}^n \ln x_i} - 1$

Note that $\hat{\theta}_{MLE}$ is a function of sufficient statistics

#97 $p(x|p) = p(1-p)^{x-1}$ $x = 1, 2, 3 \ldots$

(a) $\mu_1' = 1/p$, $m_1' = \bar{x}$, so $\hat{p}_{MOM} = 1/\bar{x}$

(b) $L(p) = p^n (1-p)^{\Sigma x_i - n}$

$\ell(p) = n \ln p + (\Sigma x_i - n) \ln(1-p)$

$$\frac{\partial \ell(p)}{\partial p} = \frac{n}{p} - \frac{\Sigma x_i - n}{1-p} = 0$$

$\Rightarrow \hat{p}_{MLE} = 1/\bar{x}$, which is the same as $\hat{p}_{MOM}$.