M440 — HW #6 — Solution

#58 $f(x) = \begin{cases} \left(\frac{2x}{\theta}\right) e^{-x^2/\theta}, & x > 0 \\ 0, & \text{o.w.} \end{cases}$

From Ex. 9.34, we know that $X_i^2 \sim Exp(\theta)$.

$L(\theta) = \underbrace{\left(\frac{2}{\theta}\right)^n}_{g(\theta)} \underbrace{\prod_{i=1}^n x_i}_{h} \underbrace{e^{-\Sigma x_i^2/\theta}}_{g(\theta, \Sigma x_i^2)}$, clearly $u = \Sigma x_i^2$ is suff. for θ.

$E\left(\sum_{i=1}^n x_i^2\right) = n\theta$

Thus, $\hat{\theta}_{MVUE} = \frac{1}{n} \sum_{i=1}^n x_i^2$

#62 $f(x|\theta) = \begin{cases} e^{-(x-\theta)} & x \geq \theta \\ 0 & \text{o.w} \end{cases}$

From Ex. 9.51, we showed that $X_{(1)}$ is suff. for θ.

then $E(X_{(1)}) = \int_\theta^\infty n x e^{-n(x-\theta)} dx = \theta + \frac{1}{n}$

Thus, $X_{(1)} - \frac{1}{n}$ is the MVUE for θ.

#63 (a) The CDF for X is $F(x) = \int_0^x \frac{3t^2}{\theta^3} dt = x^3/\theta^3$

So, pdf for $X_{(n)}$ is: $f_{X_{(n)}}(x) = n(F(x))^{n-1} f(x) = 3nx^{3n-1}/\theta^{3n}$

for $0 \leq x \leq \theta$

(b) It can be shown that $E(X_{(n)}) = \frac{3n}{3n+1}\theta$

since $X_{(n)}$ is suff for θ, $\left(\frac{3n+1}{3n} X_{(n)}\right)$ is the MVUE for θ.

#69 $f(x|\theta) = \begin{cases} (\theta+1)x^{\theta}, & 0<x<1,\ \theta>-1 \\ 0 & \text{o.w.} \end{cases}$

Step1: $\mu_1' = E(X) = \int_0^1 x(\theta+1)x^{\theta}\,dx = (\theta+1)\left.\frac{x^{\theta+2}}{\theta+2}\right|_0^1$

$= \frac{\theta+1}{\theta+2}$

Step2: $m_1' = \bar{x}$

Step3: $\mu_1' = m_1' \Rightarrow \bar{x} = \frac{\theta+1}{\theta+2}$

Step4: $\hat{\theta}_{mom} = \frac{2\bar{x}-1}{1-\bar{x}}$

So the est. for θ by the MOM is $\frac{2\bar{x}-1}{1-\bar{x}}$

~~State~~ [$\bar{X}$ is a consistent est of μ, by the Law of large Numbers, $\hat{\theta}$ converges in prob. to θ] ⟸ You can skip this.

However, this estimator is not a function of suff. stat. so it cannot be the MVUE.

#70 Since $\mu = \lambda$, the MOM estimator of λ is

$\hat{\lambda}_{mom} = \bar{X}$.

#74 (a) Step1: $\mu_1' = E(X) = \int_0^{\theta} 2x(\theta-x)/\theta^2\,dx = \theta/3$

Step2: $m_1' = \bar{x}$

Step3: $\theta/3 = \bar{x}$

Step4: $\hat{\theta}_{Mom} = 3\bar{X}$

(b) $L(\theta) = 2^n\theta^{-2n}\prod_{i=1}^{n}(\theta-x_i)$. clearly, the likelihood cannot be factored into a function that only depends on $\bar{x}$, so the MOM is not suff. for θ

#77 $X_i \overset{iid}{\sim} \text{Unif}(0, 3\theta)$.

$\mu_1' = E(X) = \frac{3}{2}\theta$

$m_1' = \bar{X}$

$\frac{3}{2}\theta = \bar{X} \Rightarrow \hat{\theta}_{MOM} = \frac{2}{3}\bar{X}$

#78 $\mu_1' = E(X) = \int_0^3 \alpha x^\alpha 3^{-\alpha} dx = \frac{3\alpha}{\alpha+1}$

$m_1' = \bar{X}$

$\frac{3\alpha}{\alpha+1} = \bar{X} \Rightarrow \hat{\theta}_{MOM} = \frac{\bar{X}}{3-\bar{X}}$

#82 (a) $L(\theta) = \theta^{-n} r^n \left(\prod_{i=1}^n x_i\right)^{r-1} \exp\left(-\sum_{i=1}^n x_i^r/\theta\right)$

$U = \sum_{i=1}^n X_i^r$ is suff stat for θ

(b) $l(\theta) = \ln(L(\theta)) = -n\ln\theta + n\ln r + (r-1)\ln\left(\prod_{i=1}^n x_i\right)$
$- \sum_{i=1}^n x_i^n/\theta$

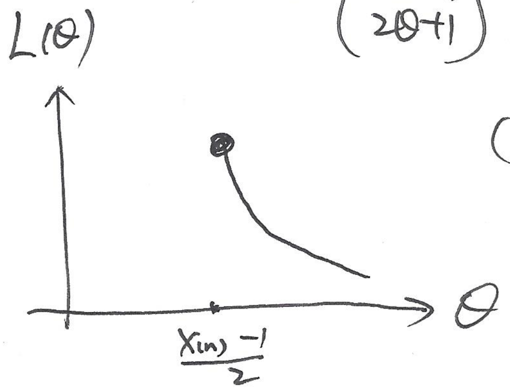
$\frac{d\, l(\theta)}{d\theta} = 0 \Rightarrow \hat{\theta}_{MLE} = \frac{1}{n}\sum_{i=1}^n X_i^r$

(c) Note that $\hat{\theta}_{MLE}$ is a function of the sufficient statistic.
Since it is easily shown that $E(X^r) = \theta$,
Then $\hat{\theta}_{MLE}$ is unbiased and the MVUE for θ.

#83 $\quad f(x|\theta) = \frac{1}{2\theta+1} I(x \le 2\theta+1)$

(a) $\mathcal{L}(\theta) = \left(\frac{1}{2\theta+1}\right)^n I(X_{(n)} \le 2\theta+1)$

$$= \left(\frac{1}{2\theta+1}\right)^n I\left(\frac{X_{(n)}-1}{2} \le \theta\right)$$



(same as class example 2)

$\hat{\theta}_{MLE} = \frac{X_{(n)}-1}{2}$ maximize $L(\theta)$.

(b) $V(X) = \frac{(2\theta+1)^2}{12}$,

by the invariance property of MLE

the MLE for $V(X) = \frac{(2\theta+1)^2}{12}$ is $\frac{(X_{(n)})^2}{12}$.

#88 $\quad L(\theta) = (\theta+1)^n \left(\prod_{i=1}^n x_i\right)^\theta$

$$\ell(\theta) = n\ln(\theta+1) + \theta\left(\ln\left(\prod_{i=1}^n x_i\right)\right)$$

$$= n\ln(\theta+1) + \theta\left(\sum_{i=1}^n \ln x_i\right)$$

$$\frac{\partial \ell(\theta)}{\partial\theta} = \frac{n}{\theta+1} + \sum_{i=1}^n \ln x_i = 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{-n}{\sum_{i=1}^n \ln x_i} - 1$$

Note that $\hat{\theta}_{MLE}$ is a function of sufficient statistics

(#97) $p(x|p) = p(1-p)^{x-1} \quad x=1,2,3\ldots$

(a) $\mu_1' = 1/p$, $m_1' = \bar{x}$, so $\hat{p}_{MOM} = 1/\bar{x}$

(b) $L(p) = p^n (1-p)^{\Sigma x_i - n}$

$\ell(p) = n \ln p + (\Sigma x_i - n) \ln(1-p)$

$$\frac{\partial \ell(p)}{\partial p} = \frac{n}{p} - \frac{\Sigma x_i - n}{1-p} = 0$$

$\Rightarrow \hat{p}_{MLE} = 1/\bar{x}$, which is the same as $\hat{p}_{MOM}$.