#7 Spz $X_1, \dots X_n \overset{iid}{\sim} f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & x > 0 \\ 0 & o\text{-}w \end{cases}$

$\hat{\theta}_1 = n X_{(1)}$, $MSE(\hat{\theta}_1) = \theta^2$,

Consider $\hat{\theta}_2 = \bar{X}$ and find eff $(\hat{\theta}_1, \hat{\theta}_2)$

Solution: $E(X_i) = \theta$, and $Var(X_i) = \theta^2$

$$MSE(\hat{\theta}_1) = Var(\hat{\theta}_1) + (B(\hat{\theta}_1))^2 = Var(\hat{\theta}_1) + \underset{\text{unbiase.}}{0} = \theta^2$$

$\Rightarrow Var(\hat{\theta}_1) = \theta^2$

Also, $Var(\hat{\theta}_2) = Var(\bar{x}) = \frac{\sigma^2}{n} = \frac{\theta^2}{n}$

Therefore, eff $(\hat{\theta}_1, \hat{\theta}_2) = \frac{\theta^2/n}{\theta^2} = 1/n$

Section 9.3 problems:

#18 Read textbook P428:

$$W = \frac{(n_1+n_2-2)S_p^2}{\sigma^2} = \frac{(n+n-2)S_p^2}{\sigma^2} = \frac{(2n-2)S_p^2}{\sigma^2} \sim \chi^2(2n-2)$$

$$S_p^2 = \frac{\Sigma(X_i-\bar{X})^2 + \Sigma(Y_i-\bar{Y})^2}{n+n-2} = \frac{\Sigma(X_i-\bar{X})^2 + \Sigma(Y_i-\bar{Y})^2}{2n-2}$$

To show S_p^2 is a consistent estimator of σ^2, we need to show:

① $E(S_p^2) = \sigma^2$: unbiase

② $\lim_{n\to\infty} Var(S_p^2) = 0$

① since $E(W) = E\left(\frac{(2n-2)S_p^2}{\sigma^2}\right) = \frac{(2n-2)}{\sigma^2} E(S_p^2) = 2n-2$

$\Rightarrow E(S_p^2) = \sigma^2$, so S_p^2 is an unbiased est. of σ^2.

② $Var(W) = Var\left(\frac{(2n-2)S_p^2}{\sigma^2}\right) = \frac{(2n-2)^2}{\sigma^4} Var(S_p^2) = 2(2n-2)$

$\Rightarrow Var(S_p^2) = \frac{2(2n-2)\sigma^4}{(2n-2)^2} = \frac{2\sigma^4}{2n-2} = \frac{\sigma^4}{n-1}$

clearly, when $n \to \infty$, $Var(S_p^2) \to 0$

That's $\lim_{n\to\infty} Var(S_p^2) = 0$

Therefore, S_p^2 is a consistent est. of σ^2.

#20 $X \sim Bin(n, p)$, show that X/n is a consistent est. of p.

We need to show: ① $E(\frac{X}{n}) = p$

② $\lim_{n\to\infty} Var(\frac{X}{n}) = 0$

Here: ① $\because X \sim Bin(n,p)$, so $E(X) = np$, $Var(X) = np(1-p)$

$E(\frac{X}{n}) = (\frac{1}{n}) E(X) = (\frac{1}{n})(np) = p$

② $Var(\frac{X}{n}) = \frac{1}{n^2} Var(X) = \frac{1}{n^2} \cdot np(1-p) = \frac{p(1-p)}{n} \to 0$

when $n \to \infty$

That's: $\lim_{n\to\infty} Var(\frac{X}{n}) = 0$

#24 Let $X_1, \ldots X_n \overset{iid}{\sim} N(0,1)$

(a) since $X_i \overset{iid}{\sim} N(0,1)$

So $X_i^2 \sim \chi^2(1)$

it follows that $\sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$

(b) Let $W_n = \frac{1}{n} \sum_{i=1}^{n} X_i^2$

$E(W_n) = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i^2\right] = \left(\frac{1}{n}\right)(n) = 1$

$Var(W_n) = \frac{1}{n^2} Var\left[\sum_{i=1}^{n} X_i^2\right] = \frac{1}{n^2}(2n) = \frac{2}{n} \to 0$

when $n \to \infty$

So, W_n converge in probability to 1.

#30 Let $X_i \overset{iid}{\sim} f(x_i) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & \text{o.w.} \end{cases}$

show that $\bar{X}$ converges in probability to some constant.

Sol: $X_i \sim Beta(\alpha = 3, \beta = 1)$

So, $E(X_i) = 3/4 = \mu$, $Var(X_i) = 3/5 = \sigma^2$

$E(\bar{X}) = \mu = 3/4$

$Var(\bar{X}) = \sigma^2/n = 3/5n \to 0$, when $n \to \infty$

So, $\bar{X}$ converge in probability to $3/4$

(#34) $f(x) = \begin{cases} \left(\frac{2x}{\theta}\right) e^{-x^2/\theta} & , x > 0 \\ 0 & , \text{o.w.} \end{cases}$

we know that $X^2 \sim Exp(\theta)$, show that $W_n = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ is a consistent estimator for θ.

Sol: To show W_n is a consistent est. for θ, we need to show

① $E(W_n) = \theta$

② $\lim_{n\to\infty} Var(W_n) = 0$

For ① $E(W_n) = \frac{1}{n} E\left(\sum_{i=1}^{n} x_i^2\right) = \frac{n\theta}{n} = \theta$

② $Var(W_n) = \theta^2/n$

Clearly, W_n is a consistent est. of θ.

Section 9.4 problems:

(#39) $x_i \overset{iid}{\sim}$ poisson(λ), show $\sum_{i=1}^{n} x_i$ is sufficient for λ

Sol: method 1 (by definition) $\longrightarrow$ Use can also use F.T. here.

$U = \sum_{i=1}^{n} x_i \sim$ poisson$(n\lambda)$ (by MGF method)

Thus, the conditional distribution is expressed as:

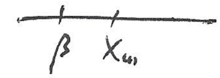
$$P(X_1 = x_1, \ldots X_n = x_n \mid U = u) = \frac{P(X_1 = x_1, \ldots X_n = x_n)}{P(U = u)}$$

$$= \frac{\prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}}{\frac{(n\lambda)^u e^{-n\lambda}}{u!}} = \frac{\frac{\lambda^{\Sigma x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!}}{\frac{n^u \lambda^u e^{-n\lambda}}{u!}} = \frac{\cancel{\lambda^u} \cancel{e^{-n\lambda}} / \prod_{i=1}^{n} x_i!}{n^u \cancel{\lambda^u} \cancel{e^{-n\lambda}} / u!} = \frac{u!}{n^u \prod_{i=1}^{n} x_i!}$$

doesn't depend on λ, Thus, $U = \Sigma x_i$ is suff. stat. for λ.

#44 $f(x|\alpha,\beta) = \alpha\beta^{\alpha}x^{-(\alpha+1)} I(x \geq \beta)$

If β is known, show that $\prod_{i=1}^{n} X_i$ is sufficient for α.



Sol. likelihood $L(\alpha) = \prod_{i=1}^{n} \alpha\beta^{\alpha}x^{-(\alpha+1)} I(x \geq \beta)$

$$= \underbrace{\alpha^{n}\beta^{n\alpha}}_{\substack{g \\ \text{parameter}}} \underbrace{\left(\prod_{i=1}^{n} X_i\right)^{-(\alpha+1)}}_{g(\prod_{i=1}^{n} X_i, \alpha)} \underbrace{I(X_{(1)} > \beta)}_{h \text{ data only}}$$

→ is given constant, not treated as a parameter

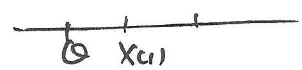
Here, $h(x_1, \dots x_n) = I(X_{(1)} > \beta)$

$$g\left(u = \prod_{i=1}^{n} X_i, \alpha\right) = \alpha^{n}\beta^{n\alpha}\left(\prod_{i=1}^{n} X_i\right)^{-(\alpha+1)}$$

clearly, it follows that $u = \prod_{i=1}^{n} X_i$ is suff. for α.

#51 $f(x|\theta) = e^{-(x-\theta)} I(x \geq \theta)$

Show that $X_{(1)} = \min(X_1, \dots X_n)$ is sufficient for θ.



Sol. $L(\theta) = \prod_{i=1}^{n} e^{-(x-\theta)} I(x \geq \theta)$

$$= e^{-\Sigma(X_i-\theta)} I(X_{(1)} \geq \theta)$$

$$= \underbrace{e^{-\Sigma X_i}}_{h} \cdot \underbrace{e^{n\theta}}_{g} \underbrace{I(X_{(1)} \geq \theta)}_{g(X_{(1)}, \theta)}$$

u.

Here, $h(x_1, \dots x_n) = e^{-\Sigma X_i}$

$$g(X_{(1)}, \theta) = e^{n\theta} I(X_{(1)} \geq \theta)$$

clearly, it follows that $u = X_{(1)}$ is suff. for θ.

(#52) $f(x|\theta) = \frac{3x^2}{\theta^3} I(x \le \theta)$

sol: $L(\theta) = \prod_{i=1}^{n} \frac{3x^2}{\theta^3} I(x \le \theta) = 3^n \cdot \underbrace{\left(\frac{1}{\theta^3}\right)^n}_{g} \underbrace{\left(\prod_{i=1}^{n} x_i^2\right)}_{h} \cdot \underbrace{\left(I(X_{(n)} < \theta)\right)}_{g(X_{(n)},\, \theta)}$

It clearly follows that $U = X_{(n)}$ is suff stat. for θ

(#54) $f(x|\alpha,\theta) = \frac{\alpha x^{\alpha-1}}{\theta^\alpha} I(x \le \theta)$

sol: $L(\alpha,\theta) = \prod_{i=1}^{n} \frac{\alpha x^{\alpha-1}}{\theta^\alpha} I(x \le \theta)$

$= \underbrace{\alpha^n}_{g} \underbrace{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}}_{g} \cdot \underbrace{\frac{1}{\theta^{n\alpha}}}_{g} I\underbrace{(X_{(n)} \le \theta)}_{g}$

parameter α only

both $U_1 = \prod_{i=1}^{n} x_i$ and parameter α, so U_1 is suff for α

parameter α and θ

both $U_2 = X_{(n)}$ and parameter θ, so U_2 is suff for θ

clearly, $g(U_1 = \prod_{i=1}^{n} x_i,\ U_2 = X_{(n)},\ \alpha, \theta) = \alpha^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \cdot \frac{1}{\theta^{n\alpha}} I(X_{(n)} \le \theta)$

$h(x_1, x_2, \ldots x_n) = 1$

So, $X_{(n)}$ and $\prod_{i=1}^{n} x_i$ are jointly suff. for α and θ.