(supplementary Ex 1):

(a) $E(\bar{X}) = E(\frac{1}{n}\Sigma X_i) = \frac{1}{n}(n\lambda) = \lambda$

(b) Recall: For large n, we have $\hat{\theta} \pm Z_{\frac{\alpha}{2}} \cdot \sigma_{\hat{\theta}}$

$$\hat{\theta} = \bar{X} = \hat{\lambda}$$

$$\sigma_{\hat{\theta}} = \sqrt{V(\bar{X})} = \sqrt{\frac{\lambda}{n}} \approx \frac{\bar{x}}{n} = \frac{\hat{\lambda}}{n}$$

$$\Rightarrow \hat{\lambda} \pm Z_{0.025} \cdot \sqrt{\frac{\bar{x}}{n}} \Rightarrow \bar{x} \pm 1.96\sqrt{\frac{\bar{x}}{n}}$$

(c) $n=30$, $\bar{x}=10$

$\Rightarrow$ 95% C.I. for λ: $\left(10 - 1.96\sqrt{\frac{10}{36}},\ 10 + 1.96\sqrt{\frac{10}{36}}\right)$

$= (8.967, 11.033)$

Section 8.8 problems

#80 The 95% CI, based on a t-distribution with $21-1=20$ DF is

$26.6 \pm 2.086(7.4/\sqrt{21})$ or 26.6 ± 3.37 or $(23.23, 29.97)$

#82 (a) The 90% C.I. for the mean Verbal SAT score for urban high school seniors is: $505 \pm 1.729(57/\sqrt{20})$ or 505 ± 22.04 or $(482.96, 527.04)$

(b) Since the interval includes the score 508, it's a plausible value for the mean

(c) The 90% CI for the mean math SAT score for urban high school seniors is: $495 \pm 1.729(69/\sqrt{20})$ or 495 ± 26.68 or $(468.32, 521.68)$

The interval does include the score 520, so the interval supports the stated true mean value.

#90

(a) For the 95% CI for the difference in mean verbal scores, the pooled sample variance is $s_p^2 = \frac{(14)(42)^2 + (14)(45)^2}{28} = 1894.5$ and

thus $446 - 534 \pm 2.048\sqrt{1894.5\left(\frac{2}{15}\right)}$

or -88 ± 32.55 or $(-120.55, -55.45)$

(b) For the 95% CI for the difference in mean math scores, the pooled sample variance is $s_p^2 = \frac{(14)(57)^2 + (14)(52)^2}{28} = 2976.5$ and

thus $548 - 517 \pm 2.048\sqrt{2976.5\left(\frac{2}{15}\right)} = 31 \pm 40.8$

or $(-9.8, 71.80)$

(c) At the 95% confidence level, there appears to be a difference in the two mean verbal SAT scores achieved by the two groups. However, a difference is not seen in the math SAT scores.

(d) We assumed that the sample measurements were independently drawn from normal populations with $\sigma_1 = \sigma_2$

#97

(a) Note that $1-\alpha = P\left(\frac{(n-1)s^2}{\sigma^2} > \chi^2_{1-\alpha, n-1}\right) = P\left(\frac{(n-1)s^2}{\chi^2_{1-\alpha, n-1}} > \sigma^2\right)$

Then, $\frac{(n-1)s^2}{\chi^2_{1-\alpha, n-1}}$ is a $100(1-\alpha)\%$ upper confidence bound for σ^2.

(b) Similar to part (a), it can be shown that $\frac{(n-1)s^2}{\chi^2_{\alpha, n-1}}$ is a $100(1-\alpha)\%$ lower confidence bound for σ^2.

#98 The confidence interval for σ^2 is $\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2, n-1}}, \frac{(n-1)s^2}{\chi^2_{1-\alpha/2, n-1}} \right)$

So since $s^2 > 0$, the confidence interval for σ is simply

$$\left(\sqrt{\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}, n-1}}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}, n-1}}} \right)$$

#102 with $n=5$, the sample variance $s^2 = 144.5$. Then

$\chi^2_{0.995} = 0.20699$ and $\chi^2_{0.005} = 14.8602$ with DF $= 4$ and the

99% CI for σ^2 is : $\left(\sqrt{\frac{(4)(144.5)}{14.8602}}, \sqrt{\frac{(4)(144.5)}{0.20699}} \right)$

or $(\sqrt{38.9}, \sqrt{2792.41})$ or $(6.2367, 52.8432)$

supplementary Ex 2.

(a) $\frac{(n_1-1)S_1^2}{\sigma_1^2} \sim \chi^2(n_1-1)$, $\frac{(n_2-1)S_2^2}{\sigma_2^2} \sim \chi^2(n_2-1)$

it's a pivot.

$$\frac{(n_1-1)S_1^2/\sigma_1^2(n_1-1)}{(n_2-1)S_2^2/\sigma_2^2(n_2-1)} = \frac{\frac{\chi^2(n_1-1)}{(n_1-1)}}{\frac{\chi^2(n_2-1)}{(n_2-1)}} = \frac{S_1^2\sigma_2^2}{S_2^2\sigma_1^2} \sim F(n_1-1, n_2-1)$$

(b). $P\left(F_{1-\frac{\alpha}{2}, n_1-1, n_2-1} < \frac{S_1^2\sigma_2^2}{S_2^2\sigma_1^2} < F_{\frac{\alpha}{2}, n_1-1, n_2-1}\right) = 1-\alpha$

So $P\left(\frac{S_2^2}{S_1^2}F_{1-\frac{\alpha}{2}, n_1-1, n_2-1} < \frac{\sigma_2^2}{\sigma_1^2} < \frac{S_2^2}{S_1^2}F_{\frac{\alpha}{2}, n_1-1, n_2-1}\right) = 1-\alpha$.

Therefore, the $100(1-\alpha)\%$ C.I. for σ_2^2/σ_1^2 is given by:

$$\left(\frac{S_2^2}{S_1^2}F_{1-\frac{\alpha}{2}, n_1-1, n_2-1}, \frac{S_2^2}{S_1^2}F_{\frac{\alpha}{2}, n_1-1, n_2-1} \right)$$

M440 — HW #5 — Solution

Section 9.2 problems:

#2 $X_1, X_2, \ldots X_n \sim (\mu, \sigma^2)$

$$\hat{\mu}_1 = \frac{1}{2}(X_1 + X_2), \quad \hat{\mu}_2 = \frac{1}{4}X_1 + \frac{X_2 + \cdots + X_{n-1}}{2(n-2)} + \frac{1}{4}X_n, \quad \hat{\mu}_3 = \bar{X}$$

(a) $E(\hat{\mu}_1) = \frac{1}{2}(\mu + \mu) = \mu \Rightarrow$ unbiased

$$E(\hat{\mu}_2) = \frac{1}{4}\mu + \frac{\mu + \cdots + \mu}{2(n-2)} + \frac{1}{4}\mu = \frac{1}{4}\mu + \frac{1}{2}\mu + \frac{1}{4}\mu = \mu \Rightarrow \text{unbiased}$$

$E(\hat{\mu}_3) = \mu \Rightarrow \hat{\mu}_3$ is an unbiased est. of μ.

(b) $Var(\hat{\mu}_1) = \frac{1}{4}(Var(X_1) + Var(X_2)) = \left(\frac{1}{4}\right)(\sigma^2 + \sigma^2) = \frac{1}{2}\sigma^2$

$$Var(\hat{\mu}_2) = \left(\frac{1}{4}\right)^2 Var(X_1) + \frac{(n-2)\sigma^2}{4(n-2)^2} + \frac{\sigma^2}{16} = \frac{\sigma^2}{8} + \frac{\sigma^2}{4(n-2)}$$

$Var(\hat{\mu}_3) = \sigma^2/n$

Thus, $\text{eff}(\hat{\mu}_3, \hat{\mu}_2) = \frac{n^2}{8(n-2)}$

and $\text{eff}(\hat{\mu}_3, \hat{\mu}_1) = n/2$

#3 $X_i \overset{iid}{\sim} \text{Unif}(\theta, \theta+1)$, $\hat{\theta}_1 = \bar{X} - \frac{1}{2}$, and $\hat{\theta}_2 = X_{(n)} - \frac{n}{n+1}$

(a) since $X_i \overset{iid}{\sim} \text{Unif}(\theta, \theta+1)$, so $E(X_i) = \frac{\theta + \theta + 1}{2} = \frac{2\theta + 1}{2} = \theta + \frac{1}{2}$

$$E(\hat{\theta}_1) = E(\bar{X} - \tfrac{1}{2}) = E(\bar{X}) - \tfrac{1}{2} = \theta + \tfrac{1}{2} - \tfrac{1}{2} = \theta$$

Thus, $\hat{\theta}_1$ is an unbiased est. of θ.

$$E(\hat{\theta}_2) = E(X_{(n)}) - \frac{n}{n+1}$$

To compute $E(X_{(n)})$, we first need to find the pdf of $X_{(n)}$.

$f_{X_{(n)}}(x) = n\,(F(x))^{n-1} f(x)$

where $F(x) = \int_{\theta}^{x} 1\, dt = x - \theta$

Therefore, $f_{X_{(n)}}(x) = n\,(x-\theta)^{n-1} \cdot 1 = n(x-\theta)^{n-1}$, for $\theta < X_{(n)} \leq \theta + 1$

consider the R.V $T = X_{(n)} - \theta$, for $0 \leq T \leq 1$

use the transformation method:

$$f_T(t) = nT^{n-1} \sim \text{Beta}(\alpha = n, \beta = 1)$$

Hence, $E(T) = \dfrac{\alpha}{\alpha+\beta} = \dfrac{n}{n+1}$

$$\text{Var}(T) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{n}{(n+1)^2(n+2)}$$

$$\Rightarrow E(X_{(n)}) = E(T+\theta) = E(T) + \theta = \frac{n}{n+1} + \theta$$

$$\Rightarrow \text{Var}(X_{(n)}) = \text{Var}(T+\theta) = \text{Var}(T) = \frac{n}{(n+1)^2(n+2)}$$

So, $E(\hat{\theta}_2) = E(X_{(n)}) - \dfrac{n}{n+1} = \dfrac{n}{n+1} + \theta - \dfrac{n}{n+1} = \theta$

clearly, $\hat{\theta}_2$ is an unbiased est. of θ.

(b) $\text{Var}(\hat{\theta}_1) = \text{Var}(\bar{x} - \frac{1}{2}) = \text{Var}(\bar{x}) = \sigma^2/n = 1/12n$

$\text{Var}(\hat{\theta}_2) = \text{Var}(X_{(n)} - \frac{n}{n+1}) = \text{Var}(X_{(n)}) = \dfrac{n}{(n+1)^2(n+2)}$

$\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \dfrac{\text{Var}(\hat{\theta}_2)}{\text{Var}(\hat{\theta}_1)} = \dfrac{12n^2}{(n+1)^2(n+2)}$

#5 spz $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\hat{\sigma}_1^2 = S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \quad \text{and} \quad \hat{\sigma}_2^2 = \frac{1}{2}(X_1 - X_2)^2$$

Solution: since $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

So, $Var\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1)$

$$\Leftrightarrow \frac{(n-1)^2}{\sigma^4} Var(S^2) = 2(n-1)$$

$$\Rightarrow Var(S^2) = Var(\hat{\sigma}_1^2) = \frac{2(n-1)\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{(n-1)}$$

Consider $\hat{\sigma}_2 = \frac{1}{2}(X_1 - X_2)^2$:

since $X_1, X_2 \sim N(\mu, \sigma^2)$,

clearly, $X_1 - X_2 \sim N(0, 2\sigma^2)$

$$\Rightarrow \frac{X_1 - X_2}{\sqrt{2}\sigma} \sim N(0, 1)$$

$$\left(\frac{X_1 - X_2}{\sqrt{2}\sigma}\right)^2 \sim \chi^2(1)$$

That's $\frac{(X_1 - X_2)^2}{2\sigma^2} \sim \chi^2(1)$

So, we have $Var\left(\frac{(X_1 - X_2)^2}{2\sigma^2}\right) = 2$

That's $\frac{1}{\sigma^4} Var\left(\frac{(X_1 - X_2)^2}{2}\right) = 2$

$$\Rightarrow Var\left(\frac{(X_1 - X_2)^2}{2}\right) = 2\sigma^4$$

Therefore, $eff(\hat{\sigma}_1^2, \hat{\sigma}_2^2) = \frac{2\sigma^4}{2\sigma^4/(n-1)} = n-1$