#39 $Y \sim \text{Gamma}(2, \beta)$

Note that, $U = \frac{2Y}{\beta} \sim \chi^2(4)$ is a pivot

$P(0.71072 \le \frac{2Y}{\beta} \le 9.48773) = 0.90$

Therefore $\left(\frac{2Y}{9.48773}, \frac{2Y}{0.71072}\right)$ forms a 90% C.I. for β.

#41 b $Y \sim N(0, \sigma^2)$,

Note that $\frac{Y^2}{\sigma^2} \sim \chi^2(1)$ is a pivot.

$P(0.0039321 \le \frac{Y^2}{\sigma^2}) = 0.95$

$\Rightarrow P(\sigma^2 \le \frac{Y^2}{0.0039321}) = 0.95$

$\Rightarrow \frac{Y^2}{0.0039321}$ is the 95% upper C.B for σ^2.

#42b $P(\frac{Y^2}{\sigma^2} \ge 0.0039321) = 0.95$

$\Rightarrow P(\sigma \le \sqrt{\frac{Y^2}{0.0039321}}) = 0.95$

$\Rightarrow \frac{Y}{0.0627}$ is the 95% upper C.B. for σ

(#44) $f_Y = \frac{2(\theta-y)}{\theta^2}$ for $0<y<\theta$

(b) show that $\frac{Y}{\theta}$ is a pivotal quantity

Pf: transformation method:

Let $U = \frac{Y}{\theta}$

Step1: $Y = \theta U$

Step2: θ

Step3: $f_U = \frac{2(\theta-\theta u)}{\theta^2} \cdot \theta = \frac{2\theta(1-u)\theta}{\theta^2} = 2(1-u)$

$U \sim \text{Beta}(\alpha=1, \beta=2)$ for $0 \le U \le 1$

Therefore, U is a pivotal quantity as it doesn't depend on any unknown parameters

(c) $P(U \le a) = 0.9$

That's $\int_0^a 2(1-u)\,du = 0.9 \Rightarrow a = 1-\sqrt{0.1} = 0.6838$

$\Rightarrow P(U \le 0.6838) = P(\frac{Y}{\theta} \le 0.6838) = P(\theta \ge \frac{Y}{0.6838}) = 0.9$

Therefore, the 90% Lower Bound for θ is $\left(\frac{Y}{0.6838}\right)$

#48 $\quad Y_1, Y_2, \ldots Y_n \overset{iid}{\sim} \text{Gamma}(\alpha = 2, \beta)$ where β is unknown.

(a) Let $U = \dfrac{2\sum_{i=1}^{n} Y_i}{\beta}$

Recall: If $U = a_1X_1 + a_2X_2 + \cdots + a_nX_n$

Then $M_U(t) = M_{X_1}(a_1t) \cdot M_{X_2}(a_2t) \cdots M_{X_n}(a_nt)$

Here, $a_1 = a_2 = \cdots = a_n = \dfrac{2}{\beta}$

Therefore, $M_U(t) = M_{Y_1}\left(\frac{2}{\beta}t\right) \cdot M_{Y_2}\left(\frac{2}{\beta}t\right) \cdots M_{Y_n}\left(\frac{2}{\beta}t\right)$

where $M_{Y_i}(t) = \left(\dfrac{1}{1-\beta t}\right)^2$

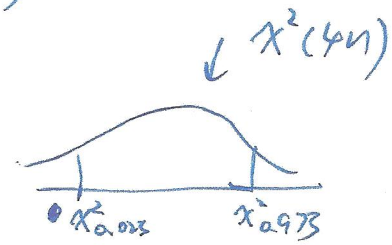
So, $M_U(t) = \left(\dfrac{1}{1-\frac{2}{\beta}\beta t}\right)^{2n} = \left(\dfrac{1}{1-2t}\right)^{2n}$

This is the MGF for the $\chi^2(4n)$

Clearly, $U = \dfrac{2\sum_{i=1}^{n} Y_i}{\beta}$ is a pivot.

(b) $P(a \le U \le b) = 0.95$ where $U \sim \chi^2(4n)$

$\left\{ P(U \le a) = 0.025 \right.$



$\Rightarrow P\left(\chi^2_{0.975}(4n) \ge U \ge \chi^2_{0.025}(4n)\right) = 0.95$

$\Rightarrow P\left(\chi^2_{0.975}(4n) \ge \dfrac{2\sum_{i=1}^{n} Y_i}{\beta} \ge \chi^2_{0.025}(4n)\right) = 0.95$

Therefore, the 95% C.I. for β is $\left(\dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{0.975}}, \dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{0.025}} \right)$

(c) C.I. is $\left(\dfrac{(2)(5)(5.39)}{34.1696}, \dfrac{(2)(5)(5.39)}{9.59083} \right)$ or $(1.577, 5.620)$

Supplementary Exercise 1

(1) $f(x|\theta) = 1$ for $\theta - \frac{1}{2} < x < \theta + \frac{1}{2}$

step 1:

Let $u = x - \theta$

① $x = u + \theta$

② 1

③ $f_U(u) = 1$ for $-\frac{1}{2} \leq u < \frac{1}{2}$

Thus, $u = x - \theta$ is a pivot

step 2:

$p(a < u < b) = 1 - \alpha$

That is
$$\begin{cases} \int_a^b 1 \, du = 1 - \alpha \\ \int_{-\frac{1}{2}}^{a} 1 \, du = \frac{\alpha}{2} \end{cases} \Rightarrow \begin{cases} b - a = 1 - \alpha \\ a + \frac{1}{2} = \frac{\alpha}{2} \end{cases} \Rightarrow \begin{cases} a = \frac{\alpha}{2} - \frac{1}{2} \\ b = \frac{1}{2} - \frac{\alpha}{2} \end{cases}$$

Thus, $p(\frac{\alpha}{2} - \frac{1}{2} < u < \frac{1}{2} - \frac{\alpha}{2}) = 1 - \alpha$

step 3: $p(\frac{\alpha}{2} - \frac{1}{2} < x - \theta < \frac{1}{2} - \frac{\alpha}{2}) = 1 - \alpha$

$p(x - \frac{1}{2} + \frac{\alpha}{2} < \theta < x - \frac{\alpha}{2} + \frac{1}{2}) = 1 - \alpha$

The $100(1-\alpha)\%$ C.I for θ is $(x - \frac{1}{2} + \frac{\alpha}{2}, \ x - \frac{\alpha}{2} + \frac{1}{2})$

(2) $f(x|\theta) = \frac{2x}{\theta^2}$ for $0 < x < \theta$ and $\theta > 0$

Step1: Let $u = \frac{x}{\theta}$.

① $x = u\theta$

② θ

③ $f_U(u) = \frac{2 \cdot u\theta}{\theta^2} \cdot \theta = 2u$ for $0 < u < 1$

Thus, $u = \frac{x}{\theta}$ is a pivot

Step2: $P(a < u < b) = 1 - \alpha$

That is
$$\begin{cases} \int_a^b 2u\,du = 1-\alpha \\ \int_0^a 2u\,du = \frac{\alpha}{2} \end{cases} \Rightarrow \begin{cases} b^2 - a^2 = 1-\alpha \\ a^2 = \frac{\alpha}{2} \end{cases} \Rightarrow \begin{cases} b = \sqrt{1-\frac{\alpha}{2}} \\ a = \sqrt{\frac{\alpha}{2}} \end{cases}$$

Thus, $P\left(\sqrt{\frac{\alpha}{2}} < \frac{x}{\theta} < \sqrt{1-\frac{\alpha}{2}}\right) = 1 - \alpha$

Step3: $P\left(\frac{x}{\sqrt{1-\frac{\alpha}{2}}} < \theta < \frac{x}{\sqrt{\frac{\alpha}{2}}}\right) = 1 - \alpha.$

The $100(1-\alpha)\%$ C.I. for θ is $\left(\frac{x}{\sqrt{1-\frac{\alpha}{2}}}, \frac{x}{\sqrt{\frac{\alpha}{2}}}\right)$.

M440_ HW #4_ Solution

Section 8.6 problems:

#56

(a) with $Z_{0.01}=2.326$, the 98% C.I. is $0.45 \pm 2.326\sqrt{\frac{(0.45)(0.55)}{800}}$

or 0.45 ± 0.041 or $(0.409, 0.491)$

(b) since the value 0.50 is not contained in the interval, there is not compelling evidence that a majority of adults in Nov, 2003 are baseball fans.

#58 The parameter of interest is μ = mean number of days required for treatment. The 95% C.I. is approximately $\bar{y} \pm Z_{0.025}(s/\sqrt{n})$

or $5.4 \pm 1.96(3.1/\sqrt{500})$ or $(5.13, 5.67)$

#61 with $Z_{0.025}=1.96$, the 95% C.I. is $167.1-140.9 \pm 1.96\sqrt{\frac{(24.3)^2+(17.6)^2}{30}}$

or $(15.46, 36.94)$

#65 (a) The 98% C.I. is, with $Z_{0.01}=2.326$ is

$$0.18-0.12 \pm 2.326\sqrt{\frac{(0.18)(0.82)+(0.12)(0.88)}{100}}$$

or 0.06 ± 0.117 or $(-0.057, 0.177)$

(supplementary Ex 1):

(a) $E(\bar{X}) = E\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n}(n\lambda) = \lambda$

(b) Recall: For large n, we have $\hat{\theta} \pm Z_{\frac{\alpha}{2}} \cdot \sigma_{\hat{\theta}}$

$$\hat{\theta} = \bar{X} = \hat{\lambda}$$

$$\sigma_{\hat{\theta}} = \sqrt{V(\bar{X})} = \sqrt{\frac{\lambda}{n}} \approx \frac{\bar{X}}{n} = \frac{\hat{\lambda}}{n}$$

$$\Rightarrow \hat{\lambda} \pm Z_{0.025} \cdot \sqrt{\frac{\bar{X}}{n}} \quad \Rightarrow \bar{X} \pm 1.96\sqrt{\frac{\bar{X}}{n}}$$

(c) $n = 30$, $\bar{x} = 10$

$\Rightarrow$ 95% C.I. for λ: $\left(10 - 1.96\sqrt{\frac{10}{36}},\ 10 + 1.96\sqrt{\frac{10}{36}}\right)$

$$= (8.967, 11.033)$$