Section 7.3 Problems

#42

(a) the population distribution is unknown. However, $n = 100 > 30$ is large sample, C.L.T. can be used here.

we can assume $\dfrac{\bar{X} - 14.5}{0.2} \sim N(0,1)$

$$P(\bar{X} > 14) = P\left(Z > \frac{14 - 14.5}{0.2}\right) = 0.0062$$

(b) $P(a < \bar{X} < b) = 0.95$

$$\Leftrightarrow P\Big(\underbrace{\boxed{\frac{a - 14.5}{0.2}}}_{= -1.96} < Z < \underbrace{\boxed{\frac{b - 14.5}{0.2}}}_{= 1.96}\Big) = 0.95$$

$$\Rightarrow \begin{cases} a = 13.608 \\ b = 14.392 \end{cases}$$

#47 $P(|\bar{X} - \mu| < 0.1) = P(-0.1 < (\bar{X} - \mu) < 0.1) = 0.9$

From #46, $n > 30$, so we can apply C.L.T. Here.

$$\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1),$$

From #46, $\sigma = \dfrac{(\text{range})}{4} = \dfrac{(8-5)}{4} = 0.75$

Here, $\dfrac{\sqrt{n}\,(0.1)}{0.75} = 1.645$

$$\Rightarrow n = 152.21$$

Therefore, 153 core samples should be taken.

#58 $X_1, X_2, \ldots X_n \sim (\mu_1, \sigma_1^2)$

$Y_1, Y_2, \ldots Y_n \sim (\mu_2, \sigma_2^2)$

Let $W = (\bar{X} - \bar{Y})$.

Note that $\begin{cases} \bar{X} \sim N(\mu_1, \frac{\sigma_1^2}{n}) \\ \bar{Y} \sim N(\mu_2, \frac{\sigma_2^2}{n}) \end{cases}$ by C.L.T. for large n.

So $\bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n})$

To standardize $(\bar{X} - \bar{Y})$, we have

$$\frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2 + \sigma_2^2}{n}}} \sim N(0,1)$$

as $n \to \infty$, by C.L.T.

#1. since $(\hat\theta-\theta)=[\hat\theta-E(\hat\theta)]+[E(\hat\theta)-\theta]$

so $(\hat\theta-\theta)^2=[\hat\theta-E(\hat\theta)]^2+2[\hat\theta-E(\hat\theta)][E(\hat\theta)-\theta]+\underbrace{[E(\hat\theta)-\theta]^2}_{B^2}$

Then $E\left[(\hat\theta-\theta)^2\right]=E\left(\underbrace{[\hat\theta-E(\hat\theta)]^2}\right)+2\underbrace{E[\hat\theta-E(\hat\theta)]}_{=0}\cdot B+B^2$

$=Var(\hat\theta)+B^2$

#6 $E(\hat\theta_1)=E(\hat\theta_2)=\theta$, $Var(\hat\theta_1)=\sigma_1^2$, $Var(\hat\theta_2)=\sigma_2^2$

$\hat\theta_3=a\hat\theta_1+(1-a)\hat\theta_2$

(a) $E(\hat\theta_3)=aE(\hat\theta_1)+(1-a)E(\hat\theta_2)=a\theta+(1-a)\theta=\theta$

so $\hat\theta_3$ is an unbiased est. for θ

(b). $Var(\hat\theta_3)=a^2Var(\hat\theta_1)+(1-a)^2Var(\hat\theta_2)$

$=a^2\sigma_1^2+(1-a)^2\sigma_2^2$ ---- (*)

To minimize $Var(\hat\theta_3)$, take derivative of (*) with respect to a

so $2a\sigma_1^2-2(1-a)\sigma_2^2=0 \Rightarrow a=\dfrac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$

#9 $f(x)=\dfrac{1}{\theta+1}e^{-x/\theta+1}$, so clearly, $X\sim Exp(\theta+1)$

$E(X)=\theta+1$,

since $E(\bar X)=\theta+1$, so $E(\bar X-1)=\theta$

Therefore, the unbiased estimator for θ is simply $(\bar X-1)$

#15 $f(y) = 3\beta^3 y^{-4}$ for $y \geq \beta$

(a) $\hat{\beta} = \min(Y_1, Y_2, \ldots Y_n) = Y_{(1)}$

The CDF of y: $F(y) = \int_\beta^y 3\beta^3 t^{-4} dt = 3\beta^3 \left(\frac{t^{-3}}{-3}\right)\Big|_\beta^y$

$= -\beta^3(y^{-3} - \beta^{-3}) = \beta^3(\beta^{-3} - y^{-3}) = 1 - \beta^3 y^{-3} = 1 - \left(\frac{\beta}{y}\right)^3$

so the p.d.f of $\hat{\beta} = Y_{(1)}$ is:

$f_{Y_{(1)}}(y) = n\,(1 - F(y))^{n-1} f(y) = n\left(\frac{\beta}{y}\right)^{3(n-1)} \cdot 3\beta^3 y^{-4}$

$= \dfrac{3n\beta^{3n}}{y^{3n+1}} = 3n\beta^{3n} y^{-(3n+1)}$

$E(\hat{\beta}) = E(Y_{(1)}) = \int_\beta^\infty y \cdot (3n) \cdot \beta^{3n} \cdot y^{-(3n+1)} dy = \left(\frac{3n}{3n-1}\right)\beta$

$\Rightarrow$ Bias $B(\hat{\beta}) = E(\hat{\beta}) - \beta = \left(\frac{1}{3n-1}\right)\beta$

(b) $MSE(\hat{\beta}) = MSE(Y_{(1)}) = Var(\hat{\beta}) + B(\hat{\beta}) = \dfrac{2}{(3n-1)(3n-2)}\beta^2$

#18 $Y_1, Y_2, \ldots Y_n \overset{iid}{\sim} Unif(0, \theta)$

so pdf of Y: $f_Y(y) = \frac{1}{\theta}$,

cdf of Y: $F_Y(y) = \dfrac{y-0}{\theta-0} = \dfrac{y}{\theta}$

$\Rightarrow$ the pdf of $Y_{(1)}$ is: $f_{Y_{(1)}}(y) = n(1 - F(y))^{n-1} f(y)$

$= n\left(1 - \frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}$

$= \frac{n}{\theta}\left(1 - \frac{y}{\theta}\right)^{n-1}$ for $0 \leq y \leq \theta$

Hence $E(Y_{(1)}) = \int_0^{\theta} y \cdot \left(\frac{n}{\theta}\right)\left(1-\frac{y}{\theta}\right)^{n-1} dy = \frac{\theta}{n+1}$

So, $E\left((n+1)\cdot Y_{(1)}\right) = \theta$

Therefore, $(n+1)Y_{(1)}$ is an unbiased est. for θ.

#19 $f(y) = \frac{1}{\theta} e^{-y/\theta} \sim Exp(\theta)$

Same technique as #15 & #18

$F(y) = 1 - e^{-y/\theta}.$

So the p.d.f. of $Y_{(1)}$ is: $f_{Y_{(1)}}(y) = n(1-F(y))^{n-1} f(y)$

$= n\, e^{-(n-1)y/\theta} \cdot \frac{1}{\theta} e^{-y/\theta}$

$\Rightarrow \cancel{E(Y_{(1)})=}$

$= \frac{n}{\theta} e^{-ny/\theta} \sim Exp\left(\frac{\theta}{n}\right)$

$\Rightarrow E(Y_{(1)}) = \frac{\theta}{n}$

$\Rightarrow E(\hat{\theta}) = E(n Y_{(1)}) = n \cdot \frac{\theta}{n} = \theta$ unbiased.

And $MSE(\hat{\theta}) = V(\hat{\theta}) + B(\hat{\theta}) = n^2 \cdot Var(Y_{(1)}) + 0 = n^2 \cdot \frac{\theta^2}{n^2} = \theta^2$

Solution — M440 — HW #3.

#23

(a) The point estimate is $\bar{y} = 11.3$ and an error bound is

$$2(16.6)/\sqrt{467} = 1.54$$

(b) The point estimate is $46.4 - 45.1 = 1.3$ and an error bound is

$$2\sqrt{\frac{(9.8)^2}{191} + \frac{(10.2)^2}{467}} = 1.7$$

(c) The point estimate is $0.78 - 0.61 = 0.17$ and an error bound is

$$2\sqrt{\frac{(0.78)(0.22)}{467} + \frac{(0.61)(0.39)}{191}} = 0.08$$

#36 $Y_1, Y_2, \dots Y_n \sim Exp(\theta)$, $E(Y_i) = \theta$, $V(Y_i) = \theta^2$,

$E(\bar{Y}) = \theta$, $Var(\bar{Y}) = \frac{\theta^2}{n}$,

(*) an unbiased est. of θ is $\bar{Y}$, since $E(\bar{Y}) = \theta$

(*) an unbiased est. of $\sqrt{\frac{\theta^2}{n}}$ is $\frac{\bar{Y}}{\sqrt{n}}$ since $E\left(\frac{\bar{Y}}{\sqrt{n}}\right) = \frac{\theta}{\sqrt{n}} = \sqrt{\frac{\theta^2}{n}}$

#38 $E(Y) = \frac{1}{p}$ and $Var(Y) = \frac{1-p}{p^2} = \frac{1}{p^2} - \frac{1}{p}$

(a) Find a function of Y that's an unbiased est. of $Var(Y)$.

Note that $E(Y) = \frac{1}{p}$ and $E(Y^2) = Var(Y) + (E(Y))^2$

$= \frac{1}{p^2} - \frac{1}{p} + \frac{1}{p^2} = \frac{2}{p^2} - \frac{1}{p}$

So the estimator $\hat{\theta}$ is a function of both Y and Y^2.

Let $\hat{\theta} = aY + bY^2$ be an unbiased est. for $\theta = \frac{1}{p^2} - \frac{1}{p}$

That's $E(\hat{\theta}) = \theta$

$$E(\hat{\theta}) = aE(Y) + bE(Y^2) = a\left(\frac{1}{p}\right) + b\left(\frac{2}{p^2} - \frac{1}{p}\right)$$

$$= (a-b)\left(\frac{1}{p}\right) + (2b)\left(\frac{1}{p^2}\right)$$

$$= -\frac{1}{p} + \frac{1}{p^2}$$

$$\Rightarrow \begin{cases} a-b = -1 \\ 2b = 1 \end{cases} \Rightarrow \begin{cases} a = -\frac{1}{2} \\ b = \frac{1}{2} \end{cases}$$

Therefore $\hat{\theta} = \frac{Y^2 - Y}{2}$ is an unbiased est. for $\theta = \frac{1}{p^2} - \frac{1}{p}$

(b) suggest how to form a 2-standard-error bound on the error of estimation when Y is used to estimate $\frac{1}{p}$

Here, Let $\theta = \frac{1}{p}$ and $\hat{\theta} = Y$

$$Var(\hat{\theta}) = Var(Y) = \frac{1}{p^2} - \frac{1}{p}$$

$$\text{2-standard-error bound} = 2\sqrt{\frac{1}{p^2} - \frac{1}{p}}$$

so, $P\left(|\hat{\theta} - \theta| \leq 2\sqrt{\frac{1}{p^2} - \frac{1}{p}}\right) = 0.95$