Math 4200 HW #2 Solution

Section 7.2. problems:

#10(a) $X_i \overset{iid}{\sim} N(\mu, 2^2)$, $i=1,2,\cdots 9$,

clearly, $\bar{X} \sim N(\mu, \frac{4}{9})$

$$\Rightarrow P(|\bar{X}-\mu| \le 0.3) = P(-0.3 \le \bar{X}-\mu \le 0.3)$$

$$= P\left(\frac{-0.3}{(\frac{2}{3})} \le \frac{\bar{X}-\mu}{(\frac{2}{3})} \le \frac{0.3}{(\frac{2}{3})}\right) = P(-0.45 \le Z \le 0.45)$$

$$= \Phi(0.45) - \Phi(-0.45) = 0.3472$$

#16 $X \sim F(p, q)$, show that $\frac{(p/q)X}{1+(p/q)X} \sim Beta(\frac{p}{2}, \frac{q}{2})$

sol: Let $u = \frac{(p/q)X}{1+(p/q)X}$, use transformation method

Step 1: $X = \frac{qu}{p(1-u)}$

Step 2: $\left|\frac{dx}{du}\right| = \frac{q}{p(1-u)^2}$

Step 3: $$f_U(u) = \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \left(\frac{p}{q}\right)^{p/2} \frac{\left(\frac{qu}{p(1-u)}\right)^{\frac{p-2}{2}}}{\left(1+\frac{p}{q}\cdot\frac{qu}{p(1-u)}\right)^{\frac{p+q}{2}}} \cdot \frac{q}{p(1-u)^2}$$

$$= \frac{u^{\frac{p}{2}-1}(1-u)^{\frac{q}{2}-1}}{B(\frac{p}{2}, \frac{q}{2})} \sim Beta\left(\frac{p}{2}, \frac{q}{2}\right)$$

#15 $X_1, X_2, \ldots X_m \overset{iid}{\sim} N(\mu_1, \sigma_1^2)$

$Y_1, Y_2, \ldots Y_n \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$

So, $\bar{X} = \frac{\Sigma X_i}{m} \sim N(\mu_1, \frac{\sigma_1^2}{m})$

$\bar{Y} = \frac{\Sigma Y_i}{n} \sim N(\mu_2, \frac{\sigma_2^2}{n})$

(a) $E(\bar{X}-\bar{Y}) = E(\bar{X}) - E(\bar{Y}) = \mu_1 - \mu_2$

(b) $V(\bar{X}-\bar{Y}) = V(\bar{X}) + V(\bar{Y}) = \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$

∵ independence between X & Y

(c) spz $\sigma_1^2 = 2$, $\sigma_2^2 = 2.5$, $m=n$.

$P(|(\bar{X}-\bar{Y})-(\mu_1-\mu_2)| \le 1) = 0.95$ ----- (*)

Consider $\bar{X}-\bar{Y} \sim N(\mu_1-\mu_2, \frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n})$

So, $Z = \frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}} \sim N(0,1)$.

Then, rewrite (*):

$P\left(Z \le \frac{1}{\sqrt{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}}\right) = P\left(Z \le \frac{1}{\sqrt{\frac{2}{n}+\frac{2.5}{n}}}\right) = 0.95.$

$\Rightarrow \frac{1}{\sqrt{\frac{4.5}{n}}} = 1.96$

$\Rightarrow n = (1.96)^2 \times 4.5 = 17.29$

#24 $n=20$, $\sigma^2=1.4$, S^2 is the sample variance of 20 measurements.

(a) Note that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(19)$

$$P(S^2 \le b) = P\left(\frac{(n-1)S^2}{\sigma^2} \le \frac{(n-1)b}{\sigma^2}\right) = P\left(\chi^2(19) \le \frac{19b}{1.4}\right) = 0.975$$

$$\Rightarrow \frac{19b}{1.4} = 32.8523 \qquad \Rightarrow b = 2.42$$

(b) $$P(a \le S^2) = P\left(\frac{(n-1)a}{\sigma^2} \le \chi^2(19)\right) = 0.975$$

$$\Rightarrow \frac{19a}{1.4} = 8.96055 \qquad \Rightarrow a = 0.656$$

(c) $$P(a \le S^2 \le b) = P\left(\frac{19a}{1.4} \le \chi^2(19) \le \frac{19b}{1.4}\right)$$

$$= P(S^2 \le b) - P(S^2 \le a) = 0.975 - (1-0.975) = 0.95$$

#26 $n=9$, σ^2 is unknown, S^2 is sample variance.

Note that (1) $\boxed{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)}$

and (2) $\boxed{\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)}$ $\Leftarrow$ we will use this one since σ is unknown.

Therefore, $$P(g_1 \le (\bar{X}-\mu) \le g_2) = P\left(\frac{g_1}{S/\sqrt{n}} \le t(n-1) \le \frac{g_2}{S/\sqrt{n}}\right)$$

$$= P\left(\frac{3g_1}{S} \le t(8) \le \frac{3g_2}{S}\right) = 0.9$$

$$\Rightarrow \begin{cases} \frac{3g_1}{S} = -1.86 \\ \frac{3g_2}{S} = 1.86 \end{cases} \Rightarrow \begin{cases} g_1 = -0.62S \\ g_2 = 0.62S \end{cases}$$

#37 $X_1, X_2, \dots X_5 \sim N(0,1)$, $\bar{X} = \frac{1}{5}\sum_{i=1}^{5} X_i$

X_6 be another independent obs. from the same population.

(a) $W = \sum_{i=1}^{5} X_i^2$

Note that $X_i \overset{iid}{\sim} N(0,1)$, then $X_i^2 \sim \chi^2(1)$.

Hence $W = \sum_{i=1}^{5} X_i^2 \sim \chi^2(5)$

(b) $U = \sum_{i=1}^{5} (X_i - \bar{X})^2$

Note that $\dfrac{\sum_{i=1}^{5} (X_i - \bar{X})^2}{\sigma^2} \sim \chi^2(n-1)$

(Remark: $\dfrac{\sum_{i=1}^{n} (X_i - \mu)^2}{\sigma^2} \sim \chi^2(n)$) $\Leftarrow$ this is a different one.

Here, $n=5$, $\sigma^2 = 1$,

Therefore. $U = \sum_{i=1}^{5} (X_i - \bar{X})^2 \sim \chi^2(4)$

(c) $\sum_{i=1}^{5} (X_i - \bar{X})^2 + X_6^2 = U + X_6^2$

Note that $\begin{cases} U \sim \chi^2(4) \\ X_6^2 \sim \chi^2(1) \end{cases}$

Hence, $\sum_{i=1}^{5} (X_i - \bar{X})^2 + X_6^2 \sim \chi^2(5)$

#38 $X_1, X_2, \ldots X_6, \bar{X}, W, U$ are defined as in #37.

(a) $$\frac{\sqrt{5}X_6}{\sqrt{W}} = \frac{X_6}{\sqrt{\frac{W}{5}}} = \frac{Z}{\sqrt{\frac{\chi^2(5)}{5}}} \sim t(5)$$

(b) $$\frac{2X_6}{\sqrt{U}} = \frac{X_6}{\sqrt{\frac{U}{4}}} = \frac{Z}{\sqrt{\frac{\chi^2(4)}{4}}} \sim t(4)$$

(c) $$\frac{2(5\bar{X}^2 + X_6^2)}{U} = \frac{5\bar{X}^2 + X_6^2}{U/2}$$

Note that $\bar{X} \sim N(0, \frac{1}{5})$, so $\frac{\bar{X}-0}{\sqrt{\frac{1}{5}}} = \sqrt{5}\bar{X} \sim N(0,1)$

Then $(\sqrt{5}\bar{X})^2 = 5\bar{X}^2 \sim \chi^2(1)$

Therefore $(5\bar{X}^2 + X_6^2) \sim \chi^2(2)$

$$\Rightarrow \frac{2(5\bar{X}^2 + X_6^2)}{U} = \frac{(5\bar{X}^2 + X_6^2)/2}{U/4} = \frac{\chi^2(2)/2}{\chi^2(4)/4} = F(2,4)$$