Math 4200 HW#1 Solution

Section 6.3 problems :

#2

$$f(y) = \begin{cases} (3/2)\,y^2 & -1 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

The distribution function of Y is : $F_Y(y) = \int_{-1}^{y} \left(\frac{3}{2}\right) t^2\,dt = \left(\frac{1}{2}\right)(y^3 + 1)$

for $-1 \le y \le 1$.

(a) $U = 3Y$,

$F_U(u) = P(U \le u) = P(3Y \le u) = P(Y \le \frac{u}{3}) = F_Y\left(\frac{u}{3}\right) = \left(\frac{1}{2}\right)\left(\frac{u^3}{18} - 1\right)$

for $-3 \le u \le 3$,

$\Rightarrow f_U(u) = (F_U(u))' = \frac{u^2}{18}$, for $-3 \le u \le 3$

#33

$$f(y) = \begin{cases} \left(\frac{3}{2}\right) y^2 + y & 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$U = 5 - (Y/2)$

step1 : $g^{-1}(u) = Y = 2(5-u) = 10 - 2u$

step2 : $\frac{dg^{-1}(u)}{du} = -2$

step3 : $f_U(u) = \left[\left(\frac{3}{2}\right)(10-2u)^2 + (10-2u)\right]|-2|$

$= 4(80 - 31u + 3u^2)$ for $4.5 \le u \le 5$

#46 $Y \sim \text{Gamma}(\frac{n}{2}, \beta)$, $W = \frac{2Y}{\beta}$

show $W \sim \chi^2(n)$

$\therefore E[e^{(\frac{2t}{\beta})Y}]$

Pf: $M_W(t) = E[e^{Wt}] = E[e^{2Yt/\beta}] = E[e^{(\frac{2Y}{\beta})t}] = M_Y(\frac{2t}{\beta})$

since $Y \sim \text{Gamma}(\frac{n}{2}, \beta)$,

then $E_Y(t) = \left(\frac{1}{1-\beta t}\right)^{\frac{n}{2}}$

$\Rightarrow M_W(t) = M_Y(\frac{2t}{\beta}) = \left(\frac{1}{1-\beta \cdot \frac{2t}{\beta}}\right)^{\frac{n}{2}} = \left(\frac{1}{1-2t}\right)^{n/2}$

$\Rightarrow W \sim \chi^2(n)$

$\boxed{\#6.74}$ Since $Y_i \overset{iid}{\sim} \text{Unif}(0,\theta)$, then $f_{Y_i}(y_i) = \frac{1}{\theta}$ for $0 \le y_i \le \theta$.

and $F_{Y_i}(y_i) = y/\theta$ for $0 \le y \le \theta$.

(a). $F_{Y_{(n)}}(y) = \left(\frac{y}{\theta}\right)^n$, for $0 \le y \le \theta$

(b) $f_{Y_{(n)}}(y) = ny^{n-1}/\theta^n$, for $0 \le y \le \theta$

(c). $E(Y_{(n)}) = \int_0^\theta (y)(ny^{n-1}/\theta^n)\,dy = \left(\frac{n}{n+1}\right)\theta$

$V(Y_{(n)}) = \frac{n\theta^2}{(n+1)^2(n+2)}$.

use $E(Y_{(n)}^2) - (E(Y_{(n)}))^2$.

#1 $Z \sim N(0,1)$, let $U = Z^2$, show $U \sim \chi^2(1)$

Pf: Use MGF method:

$$M_U(t) = E[e^{ut}] = E[e^{z^2 t}] = \int_{-\infty}^{\infty} e^{z^2 \cdot t} \cdot f_Z(z)\, dz$$

$$= \int_{-\infty}^{\infty} e^{z^2 t} \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(t-\frac{1}{2})z^2}\, dz \quad \cdots\cdots (*)$$

Recall: If $\forall X \sim N(0, \sigma^2)$, then $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$

So, $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx = 1$

$\Leftrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx = \sigma \quad (**)$

Rewrite $(*)$, we have:

$$(*) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2(1-2t)}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2(\frac{1}{1-2t})}} dz$$

$$\underset{\substack{\text{According to} \\ (**),\ \text{Here } \sigma^2 = \frac{1}{1-2t}}}{=} \left(\frac{1}{1-2t}\right)^{\frac{1}{2}}$$ is MGF for $\chi^2(1)$

Therefore, $U = Z^2 \sim \chi^2(1)$.

#2 $M_Y(t) = E(e^{Yt}) = E(e^{\frac{2Xt}{\beta}}) = M_X\left(\frac{2t}{\beta}\right) = \left[\frac{1}{1-\beta\cdot\left(\frac{2t}{\beta}\right)}\right]^{\alpha} = \left(\frac{1}{1-2t}\right)^{\alpha}$

this is the MGF for $\chi^2(2\alpha)$.

#3 Let $U \sim unif(0,1)$, Define $X = -\log U$ and $Y = -\log(1-U)$.

(a). transformation.

step1: $U = e^{-x}$

step2: $\frac{du}{dx} = -e^{-x}$

step3: $f_X(x) = \frac{1}{1-0} e^{-x} = e^{-x} \sim Exp(1)$

(b) transformation

step1: $U = 1 - e^{-Y}$

step2: $\frac{du}{dy} = e^{-Y}$

step3: $f_Y(y) = \frac{1}{1-0} \cdot e^{-Y} = e^{-Y} \sim Exp(1)$.

#4 (a) Given $X \sim Gamma(\alpha_1, \beta)$, $Y \sim Gamma(\alpha_2, \beta)$

$U = X + Y$

The MGF of U: $M_U(t) = M_X(t) \cdot M_Y(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha_1} \cdot \left(\frac{1}{1-\beta t}\right)^{\alpha_2} = \left(\frac{1}{1-\beta t}\right)^{\alpha_1+\alpha_2}$

this is the MGF for Gamma $(\alpha_1+\alpha_2, \beta)$

Therefore: $U \sim Gamma(\alpha_1+\alpha_2, \beta)$

(b) Find the pdf of $\frac{X+Y}{\beta}$

Let the R.V. $T = \frac{X+Y}{\beta} = \frac{U}{\beta}$ (U is a linear monotonic transformation of U, Hence transformation is reasonable)

step1: $U = \beta T$

step2: $\frac{du}{dT} = \beta$

step3: $g_U^{-1}(u) \cdot \left|\frac{du}{dT}\right| = \frac{(\beta T)^{\alpha_1+\alpha_2-1} e^{-\beta T/\beta}}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} \cdot \beta = \frac{T^{\alpha_1+\alpha_2-1} e^{-T}}{\Gamma(\alpha_1+\alpha_2)} \sim Gamma(\alpha_1+\alpha_2, 1)$

Therefore, $f_T(t) = \frac{t^{\alpha_1+\alpha_2-1} e^{-t}}{\Gamma(\alpha_1+\alpha_2)}$, which is Gamma$(\alpha_1+\alpha_2, 1)$