M440 – SA #4 – Solution

#1 $X_i \overset{iid}{\sim} \text{Unif}(0,\theta)$, Let $\hat{\theta}_1 = \frac{n+1}{n} X_{(n)}$ and $\hat{\theta}_2 = 3\bar{X}$

(a) Refer to textbook p446 Example 9.1

$$E(\hat{\theta}_1) = \frac{n+1}{n} E(X_{(n)}) = \left(\frac{n+1}{n}\right)\left(\frac{n}{n+1}\right)\theta = \theta$$

Hence, $\hat{\theta}_1$ is an unbiased est. of θ.

(b) $E(\hat{\theta}_2) = 3E(\bar{X}) = 3\left(\frac{\theta}{2}\right) = \frac{3\theta}{2}$ is biased

However, $\hat{\theta}_{new2} = \frac{2}{3}\hat{\theta}_2 = \left(\frac{2}{3}\right)(3\bar{X}) = 2\bar{X}$ is an unbiased est.

And we'll use $\frac{2}{3}\hat{\theta}_2$ for part (c)

(c). $$\text{eff}(\hat{\theta}_2, \hat{\theta}_1) = \frac{Var(\hat{\theta}_1)}{Var(\hat{\theta}_{new2})} = \frac{\frac{\theta^2}{n(n+2)}}{\frac{\theta^2}{3n}} = \frac{3}{n+2}$$

#2 Let $x_i \overset{iid}{\sim} f(x|\theta) = \theta x^{\theta-1}$ for $0 \le x \le 1$

<u>Step1:</u> show $\bar{X}$ is an unbiased est. for $\frac{\theta}{\theta+1}$

<u>Reason:</u> $X_i \sim \text{Beta}(\alpha=\theta, \beta=1)$.

So, $E(\bar{X}) = E(X) = \frac{\alpha}{\alpha+\beta} = \frac{\theta}{\theta+1}$, and $Var(X) = \frac{\theta}{(\theta+1)^2(\theta+2)}$

Hence. $\bar{X}$ is unbiased for $\frac{\theta}{\theta+1}$

<u>Step2:</u> Find $Var(\bar{X}) = \frac{\sigma^2}{n} = \frac{\theta}{n(\theta+1)^2(\theta+2)} \to 0$

when $n \to \infty$

That's, $\lim_{n\to\infty} Var(\bar{X}) = 0$.

Therefore, $\bar{X}$ is a consistent estimator of $\frac{\theta}{\theta+1}$

(#3) $X_i \sim f(x_i|\alpha,\beta) = \alpha\beta^{\alpha} x^{-(\alpha-1)}$ for $x \geq \beta$

(a) $\mathcal{L}(\alpha,\beta) = \alpha^n \beta^{n\alpha} \left(\prod_{i=1}^{n} x_i\right)^{-(\alpha-1)} I(X_{(1)} \geq \beta)$

(b) Base on the likelihood function on part (a),

$$\mathcal{L}(\alpha,\beta) = \underbrace{\alpha^n}_{h} \underbrace{\beta^{n\alpha}}_{g} \underbrace{\left(\prod_{i=1}^{n} x_i\right)^{-(\alpha-1)}}_{h} \cdot \underbrace{I(X_{(1)} \geq \beta)}_{g}$$

where $h(x_1, x_2, \dots x_n) = \alpha^n \left(\prod_{i=1}^{n} x_i\right)^{-(\alpha-1)}$

~~and~~ $g(\beta, u) = \beta^{n\alpha} I(x_{(1)} \geq \beta)$ and $u = X_{(1)}$.

Therefore, $X_{(1)}$ is sufficient for β, if α is known.

(c) If α and β are both unknown parameter, then

$$\mathcal{L}(\alpha,\beta) = \underbrace{\alpha^n}_{h(\alpha)} \underbrace{\beta^{n\alpha}}_{h(\beta)} \underbrace{\left(\prod_{i=1}^{n} x_i\right)^{-(\alpha-1)}}_{h(\prod_{i=1}^{n} x_i, \alpha)} \cdot \underbrace{I(X_{(1)} \geq \beta)}_{h(X_{(1)}, \beta)}$$

So, $h(\alpha, \beta, \prod_{i=1}^{n} x_i, X_{(1)}) = \alpha^n \left(\prod_{i=1}^{n} x_i\right)^{-(\alpha-1)} \cdot \beta^{n\alpha} I(X_{(1)} \geq \beta)$.

Therefore, $\prod_{i=1}^{n} X_i$ and $X_{(1)}$ are joint sufficient stat for α and β.

#4 $X_i \overset{iid}{\sim} N(\mu, 1)$

(a) Step1: Find suff stat for μ by FT.

$$L(\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x_i-\mu)^2/2} = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\Sigma(x_i-\mu)^2/2}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}(\Sigma(x_i^2-2\mu x_i+\mu^2))} = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}(\Sigma x_i^2-2\mu\Sigma x_i+n\mu^2)}$$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi}}\right)^n}_{h} \underbrace{e^{-\frac{1}{2}\Sigma x_i^2}}_{h} \cdot \underbrace{e^{\mu n\bar{x}} \cdot e^{-\frac{1}{2}n\mu^2}}_{g(\bar{x},\mu)}$$

Thus, ~~the~~ $\bar{X}$ is suff stat. for μ.

(Note that: $\sum_{i=1}^{n} x_i$ is suff stat for μ as well. You can use either one. They both lead to the same answer).

Step2: Note that $E(\bar{X}) = \mu$ and $Var(\bar{X}) = \frac{\sigma^2}{n} = 1/n$

Consider $E(\bar{X}^2) = 1/n + \mu^2$, that's $E(\bar{X}^2 - 1/n) = \mu^2$

So, $\phi(T(X)) = \bar{X}^2 - 1/n$.

(b). $Var(\hat{\mu}^2) = Var(\bar{X}^2 - 1/n) = Var(\bar{X}^2) = E(\bar{X}^4) - (E(\bar{X}^2))^2$

$$= E(\bar{X}^4) - \left(\tfrac{1}{n}+\mu^2\right)^2 = \underbrace{\left(\frac{3}{n^2} + \frac{6\mu^2}{n} + \mu^4\right)}_{\text{use MGF}} - \left(\tfrac{1}{n}+\mu^2\right)^2$$

$$= (2+4n\mu^2)/n^2$$

#5 $f(x|\theta) = \frac{1}{(r-1)!\,\theta^r} e^{-x/\theta} x^{r-1}$, $x > 0$

$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = [(r-1)!]^{-n} \theta^{-nr} e^{-\frac{\Sigma x_i}{\theta}} \left(\prod_{i=1}^{n} x_i\right)^{r-1}$

FT.

$T(x) = \sum_{i=1}^{n} x_i$ is a suff stat for θ.

$E(T(x)) = E(x_1 + x_2 + \cdots + x_n) = nr\theta$

$\Rightarrow E\left(\frac{T(x)}{nr}\right) = \theta$

Thus, $\phi(T(x)) = \frac{T(x)}{nr} = \frac{\Sigma x_i}{nr}$ is MVUE for θ.

$\theta = \frac{\bar{x}}{r}$

#5 $x_i \sim \text{pois}(\lambda)$

(a) $p(x_i) = \dfrac{\lambda^{x_i} e^{-\lambda}}{x_i!}$

Likelihood $L(\lambda) = \prod_{i=1}^{n} \left(\dfrac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right) = \dfrac{\lambda^{\Sigma x_i} e^{-n\lambda}}{\prod_{i=1}^{n}(x_i!)}$

Then, $\ell(\lambda) = \ln(L(\lambda)) = \left(\sum_{i=1}^{n} x_i\right)\ln\lambda - n\lambda - \ln\left(\prod_{i=1}^{n} x_i!\right)$

$\dfrac{\partial \ell(\lambda)}{\partial \lambda} = \dfrac{\sum_{i=1}^{n} x_i}{\lambda} - n = 0 \quad \Rightarrow \hat{\lambda}_{MLE} = \bar{x}$

(b) By the invariance property,

$Var(x) = \lambda$ and hence the std deviation of x is $\sqrt{\lambda}$

$\widehat{(\sqrt{\lambda})}_{MLE} = \sqrt{\bar{x}}$

#6 $x_i \sim f(x|\theta) = \frac{1}{\theta} e^{-x/\theta} \sim \text{Exp}(\theta)$

(a) step 1: $E(x) = \theta = \mu_1'$

step 2: $m_1' = \bar{x}$

step 3: $\theta = \bar{x}$

step 4: $\hat{\theta}_{mom} = \bar{x}$

(b) $L(\theta) = \left(\frac{1}{\theta}\right)^n e^{-\Sigma x_i/\theta} = (\theta^{-n}) e^{-\Sigma x_i/\theta}$

$\ell(\theta) = \ln(L(\theta)) = (-n)\ln\theta - \dfrac{\Sigma x_i}{\theta}$

$\dfrac{\partial \ell(\theta)}{\partial \theta} = \dfrac{-n}{\theta} + \dfrac{\Sigma x_i}{\theta^2} = 0$

$\Rightarrow \hat{\theta}_{MLE} = \bar{x}$

(c) $P(Y \leq 2) = \int_0^2 \frac{1}{\theta} e^{-x/\theta} dx = 1 - e^{-2/\theta}$

By the invariance property, $1 - e^{-2/\bar{x}}$ is the MLE for $P(Y \leq 2)$

(d) Let x_m be the median such that

$$\int_0^{x_m} \frac{1}{\theta} e^{-x/\theta} dx = 1/2$$

$$\Rightarrow x_m = \ln(2)\theta$$

By the invariance property, $(\ln 2)(\bar{x})$ is the MLE for the median.