Solution — M440 — SA #2

#1 $Y \sim Bin(n, p)$, $E(Y) = np$, $Var(Y) = np(1-p)$

(a) show $n\left(\frac{Y}{n}\right)\left(1-\frac{Y}{n}\right)$ is an unbiased est. of $Var(Y) = np(1-p)$

Pf: Let $\hat{\theta} = n\left(\frac{Y}{n}\right)\left(1-\frac{Y}{n}\right)$

$$E(\hat{\theta}) = E\left[n\left(\frac{Y}{n}\right)\left(1-\frac{Y}{n}\right)\right] = nE\left[\frac{Y}{n} - \frac{Y^2}{n^2}\right] = (n)E\left[\frac{Y}{n}\right] - (n)\cdot E\left[\frac{Y^2}{n^2}\right]$$

$$= (n)\left(\frac{1}{n}\right)E(Y) - (n)\left(\frac{1}{n^2}\right)E(Y^2)$$

$$= E(Y) - E(Y^2)\cdot\left(\frac{1}{n}\right)$$

Note that $Var(Y) = E(Y^2) - (E(Y))^2$

So, $E(Y^2) = Var(Y) + (E(Y))^2 = np(1-p) + n^2p^2 = np - np^2 + n^2p^2$

Therefore, $E(\hat{\theta}) = E(Y) - E(Y^2)\cdot\left(\frac{1}{n}\right) = (np) - (np - np^2 + n^2p^2)\left(\frac{1}{n}\right)$

$$= np - p + p^2 - np^2$$

(b). Form an unbiased est. of $Var(Y)$

$\because$ $E(\hat{\theta}) = np - p + p^2 - np^2 = p(n-1) - p^2(n-1) = (n-1)(p)(1-p)$

$\theta = np - np^2 = np(1-p)$

So consider the estimator $\hat{\theta} = \left(\frac{n}{n-1}\right)(n)\left(\frac{Y}{n}\right)\left(1-\frac{Y}{n}\right)$

#2 $Y_1, Y_2, \dots Y_n \sim f(y) = \dfrac{\alpha y^{\alpha-1}}{\theta^\alpha}$ for $0 \le y \le \theta$. α is known, θ is unknown.

$\hat{\theta} = Y_{(n)} = \max(Y_1, Y_2, \dots Y_n)$

(a) show that $\hat{\theta}$ is a biased est. for θ.

Pf: The pdf of $\hat{\theta} = Y_{(n)}$ is:

$$f_{Y_{(n)}}(y) = n\,(F(y))^{n-1} f(y)$$

$$F(y) = \int_0^y \frac{\alpha t^{\alpha-1}}{\theta^\alpha}\,dt = \left(\frac{\alpha}{\theta^\alpha}\right)\left(\frac{1}{\alpha}\right) t^\alpha \Big|_0^y = \left(\frac{1}{\theta^\alpha}\right)(y^\alpha) = \left(\frac{y}{\theta}\right)^\alpha$$

so, $$f_{Y_{(n)}}(y) = n \cdot \left(\frac{y}{\theta}\right)^{\alpha(n-1)} \cdot \frac{\alpha}{\theta^\alpha} \cdot y^{\alpha-1}$$

$$= (n\alpha)\left(\frac{1}{\theta}\right)^{\alpha n} (y)^{\alpha n-1} \quad \text{for } 0 \le y_{(n)} \le \theta$$

Then $$E(Y_{(n)}) = \int_0^\theta (n\alpha)\left(\frac{1}{\theta}\right)^{\alpha n} \cdot (y)\,(y)^{\alpha n-1}\,dy$$

$$= \frac{n\alpha}{\theta^{n\alpha}} \int_0^\theta y^{\alpha n}\,dy$$

$$= \left(\frac{n\alpha}{\theta^{n\alpha}}\right) \cdot \frac{y^{\alpha n+1}}{\alpha n+1}\Big|_0^\theta = \frac{n\alpha\,\theta^{\alpha n+1}}{\theta^{n\alpha}(\alpha n+1)}$$

$$= \left(\frac{n\alpha}{\alpha n+1}\right)\theta$$

Hence $\hat{\theta} = Y_{(n)}$ is a biased est. for θ.

(b) consider $\boxed{\left(\frac{n\alpha+1}{n\alpha}\right) Y_{(n)}}$ as an unbiased est. of θ

(c) $MSE(\hat{\theta}) = E[(Y_{(n)} - \theta)^2] = E[Y_{(n)}^2] - 2\theta E(Y_{(n)}) + \theta^2$

$$E(Y_{(n)}^2) = \int_0^\theta (n\alpha)\left(\frac{1}{\theta}\right)^{n\alpha} (y^2)(y)^{\alpha n-1}\, dy$$

$$= \frac{n\alpha}{\theta^{n\alpha}} \int_0^\theta y^{\alpha n+1}\, dy = \frac{n\alpha}{\theta^{n\alpha}} y^{\alpha n+2}\Big|_0^\theta$$

$$= \frac{n\alpha}{\theta^{n\alpha}} \theta^{\alpha n+2} = n\alpha\theta^2$$

Hence, $MSE(\hat{\theta}) = n\alpha\theta^2 - 2\theta\left(\frac{n\alpha}{\alpha n+1}\right)\theta + \theta^2$

$$= \frac{2}{(n\alpha+1)(n\alpha+2)}\theta^2$$

#3 Spz $Y_1, Y_2, \ldots Y_n \sim \text{unif}(\theta - \frac{1}{2}, \theta + \frac{1}{2})$ $\begin{cases} E(Y_i) = \frac{1}{2}(2\theta) = \theta \\ V(Y_i) = \frac{1}{12} \end{cases}$

Let $\hat{\theta}_1 = \frac{Y_1 + Y_2}{2}$ and $\hat{\theta}_2 = \bar{Y}$

solution: $MSE(\hat{\theta}_1) = Var(\hat{\theta}_1) + (B(\hat{\theta}_1))^2 = Var(\hat{\theta}_1) + [E(\hat{\theta}_1) - \theta]^2$

$$E(\hat{\theta}_1) = E\left(\frac{Y_1 + Y_2}{2}\right) = \frac{\theta + \theta}{2} = \theta$$

$$Var(\hat{\theta}_1) = \frac{1}{4}\left(\frac{1}{12} + \frac{1}{12}\right) = \frac{1}{24}$$

$$\Rightarrow MSE(\hat{\theta}_1) = \left(\frac{1}{24}\right) + (\theta - \theta)^2 = \frac{1}{24}$$

$$MSE(\hat{\theta}_2) = Var(\hat{\theta}_2) + [E(\hat{\theta}_2) - \theta]^2$$

$$E(\hat{\theta}_2) = E(\bar{Y}) = \frac{n\theta}{n} = \theta$$

$$Var(\hat{\theta}_2) = \frac{\sigma^2}{n} = \frac{1}{12n}$$

$$\Rightarrow MSE(\hat{\theta}_2) = \frac{1}{12n} + (\theta - \theta)^2 = \frac{1}{12n}$$

Therefore, if $n > 1$, then $\hat{\theta}_2$ is a better estimator since it has a lower MSE.

#14

Step1: $V = \frac{2X}{\theta} \sim \chi^2(2)$

So $V = \frac{2X}{\theta}$ is a pivot as it doesn't depend on unknown parameters

Step2: $P(a \le \frac{2X}{\theta} \le b) = 0.90$

Look for the $\chi^2(2)$ table, we can easily get

$$\begin{cases} a = 0.102587 \\ b = 5.99147 \end{cases}$$

$\Rightarrow \quad P(0.102587 \le \frac{2X}{\theta} \le 5.99147) = 0.9$

Step3:

$\Rightarrow \quad \left(\frac{2Y}{5.99147}, \frac{2Y}{0.102587}\right)$ represents a 90% C.I. for θ.

#5 $X_1, X_2 \sim Unif(0, \theta)$, $R = X_{(2)} - X_{(1)}$

$f_R(r) = \left(\frac{2}{\theta^2}\right)(\theta - r)$, for $0 < r < \theta$

(a) Let $U = \frac{R}{\theta}$

Use transformation method to find the pdf of U, $f_U(u)$

step1: $R = U\theta$

step2: $\frac{dR}{dU} = \theta$

step3: $f_U = f_R(u\theta) \cdot \theta = \left(\frac{2}{\theta^2}\right)(\theta - u\theta) \cdot \theta = \left(\frac{2}{\theta^2}\right)(\theta)(1-u)\theta = 2(1-u)$

$\Rightarrow f_U(u) = 2(1-u)$, for $0 < u < 1$

This is $Beta(\alpha = 1, \beta = 2)$

That's $U = \frac{R}{\theta} \sim Beta(1, 2)$, which is a pivot as it doesn't depend on any unknown parameters.

(b) $P(a \leq \frac{R}{\theta} \leq b) = 0.9$

$$\Rightarrow \begin{cases} \int_a^b 2(1-u)\,du = 0.9 \\ \int_0^a 2(1-u)\,du = 0.05 \end{cases} \Rightarrow \begin{cases} a = 0.025 \\ b = 0.776 \end{cases}$$

Hence, the 90% C.I. for θ is $\left(\frac{R}{0.776}, \frac{R}{0.025}\right)$

That's $\left(\frac{X_{(2)} - X_{(1)}}{0.776}, \frac{X_{(2)} - X_{(1)}}{0.025}\right)$

~~#3~~ #6 $X_i \overset{iid}{\sim} N(\theta, \theta)$ where $\theta > 0$

choices of pivots (1) $U_1 = \dfrac{\bar{X} - \theta}{\sqrt{\theta}/\sqrt{n}} \sim N(0,1)$

(2) $U_2 = \dfrac{\bar{X} - \theta}{S/\sqrt{n}} \sim t(n-1)$

(3) $U_3 = \dfrac{(n-1)S^2}{\theta} \sim \chi^2(n-1)$

U_2 & U_3 are easier than U_1.

Take $U_2 = \dfrac{\bar{X} - \theta}{S/\sqrt{n}}$ as an example.

$$P\left(-t_{\frac{\alpha}{2},n-1} < \frac{\bar{X} - \theta}{S/\sqrt{n}} < t_{\frac{\alpha}{2},n-1}\right) = P\left(\bar{X} - t_{\frac{\alpha}{2},n-1}\left(\frac{S}{\sqrt{n}}\right) < \theta < \bar{X} + t_{\frac{\alpha}{2},n-1}\left(\frac{S}{\sqrt{n}}\right)\right)$$
$$= 1 - \alpha.$$

The C.I for θ is: $\left(\bar{X} \pm t_{\frac{\alpha}{2},n-1} \cdot \frac{S}{\sqrt{n}}\right)$.

#4 #7 Let $X \sim Beta(\theta, 1)$, where θ is unknown

(1) Let $u_1 = -\frac{1}{\log x}$ Then C.I. for θ is $[\frac{u_1}{2}, u_1]$

$\because X \sim Beta(\theta, 1)$, $f_X(x) = \frac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma(1)} \cdot x^{\theta-1}(1-x)^0 = \theta x^{\theta-1}$

use transformation to find pdf of u_1:

① $x = e^{-\frac{1}{u_1}}$ ② $\frac{1}{u_1^2} e^{-\frac{1}{u_1}}$ ③ $f_{u_1}(u) = \frac{\theta}{u^2} e^{-\frac{\theta}{u}}$ for $0 < u < \infty$

Given that the $100(1-\alpha)\%$ C.I. for θ is $[\frac{u_1}{2}, u_1]$

that is. $P(\frac{u_1}{2} < \theta < u_1) = 1-\alpha$

or $P(\theta < u_1 < 2\theta) = 1-\alpha$

$\Rightarrow \int_\theta^{2\theta} \frac{\theta}{u^2} \cdot e^{-\frac{\theta}{u}} du = e^{-\theta/u} \Big|_\theta^{2\theta} = e^{-\frac{1}{2}} - e^{-1} = 0.239$

Thus, the confidence coefficient $1-\alpha = 0.239$

(2) Let $u_2 = x^\theta$

use transformation to find pdf of u_2:

① $x = u_2^{\frac{1}{\theta}}$ ② $(\frac{1}{\theta}) u_2^{(\frac{1}{\theta}-1)}$ ③ $f_{u_2}(u) = \theta (u_2^{\frac{1}{\theta}})^{\theta-1} \cdot \frac{1}{\theta} \cdot u_2^{\frac{1}{\theta}-1}$

$\Rightarrow f_{u_2}(u) = 1$ for $0 < u_2 < 1$

$u_2 \sim unif(0,1)$ ~~for~~ $\Rightarrow u_2$ is a pivot.

(3) $P(a < x^\theta < b) = 0.239 \Rightarrow \begin{cases} \int_a^b 1\, du = 0.239 \\ \int_0^a 1\, du = 0.3805 \end{cases}$

$\Rightarrow \begin{cases} b-a = 0.239 \\ a = 0.3805 \end{cases} \Rightarrow \begin{cases} a = 0.3805 \\ b = 0.6195 \end{cases} \Rightarrow (\log 0.3805, \log 0.6195)$

#5 #8 $X_i \overset{iid}{\sim} Unif(0, \theta)$

(1) see previous practice.

$f_U(u) = nu^{n-1}$ for $0 < u < 1$ is a pivot.

(2) $P(a < u < b) = 1-\alpha$

$\Leftrightarrow P\left(a < \frac{X_{(n)}}{\theta} < b\right) = P\left(\frac{X_{(n)}}{b} < \theta < \frac{X_{(n)}}{a}\right) = 1-\alpha$

The length of the ~~the~~ C.I for θ is $d = \frac{X_{(n)}}{a} - \frac{X_{(n)}}{b} = \frac{X_{(n)}(b-a)}{ab}$

Since $f_U(u)$ is increasing, $(b-a)$ is minimized and ab is maximized
이 $b=1$.

Thus shortest interval will have $b=1$ and a satisfying

$\alpha = \int_0^a nu^{n-1} du = a^n \Rightarrow a = \alpha^{\frac{1}{n}}$

Therefore, the shortest $(1-\alpha)100\%$ C.I. for θ is $\left(X_{(n)}, \frac{X_{(n)}}{\alpha^{\frac{1}{n}}}\right)$

~~Problem~~ #9: $X_i \overset{iid}{\sim}$ Poisson(λ) $E(X_i)=\lambda$, $V(X_i)=\lambda$.

(1) suggest $\hat{\lambda}=\bar{X}$, since $E(\bar{X})=E(X_i)=\lambda$.

(2) Try $E(\bar{X}^2)=V(\bar{X})+[E(\bar{X})]^2=\frac{\lambda}{n}+\lambda^2$

Let $\hat{\theta}=a\bar{X}^2+b\bar{X}$ be an unbiased est. for $(5\lambda-2\lambda^2)$

such that $E(\hat{\theta})=5\lambda-2\lambda^2$

That's $aE(\bar{X}^2)+bE(\bar{X})=\frac{a\lambda}{n}+a\lambda^2+b\lambda=a\lambda^2+(\frac{a}{n}+b)\lambda$

$=-2\lambda^2+5\lambda$

$$\begin{cases} a=-2 \\ \frac{a}{n}+b=5 \end{cases} \Rightarrow \begin{cases} a=-2 \\ b=5+\frac{2}{n} \end{cases}$$

suggest $-2\bar{X}^2+(5+\frac{2}{n})\bar{X}$ as the unbiased est. for $5\lambda-2\lambda^2$.

~~Problem~~ #10:

The point estimate is given by the difference of the sample proportions: $\frac{126}{180}-\frac{54}{100}=0.70-0.54=0.16$

The error bound B: $2\sqrt{\frac{(0.7)(0.3)}{180}+\frac{(0.54)(0.46)}{100}}=0.121$

$P(|\hat{p}-p|<0.121)=P(|Z|<\frac{0.121-0}{0.121})=P(|Z|<1)=0.68$