Math 4200 Student Activity #2
Point Estimators and Interval Estimations

1. We have seen that if Y has a binomial distribution with parameters n and p, then Y/n is an unbiased estimator of p. To estimate the variance of Y, we generally use n(Y/n)(1-Y/n).

a) Show that the suggested estimator is a biased estimator of Var (Y).
b) Modify n(Y/n)(1-Y/n) slightly to form an unbiased estimator of Var(Y).

2. Let $Y_1, Y_2, \ldots Y_n$ denote a random sample of size n from a population whose density is given by

$$f(y) = \begin{cases} \dfrac{\alpha y^{\alpha-1}}{\theta^{\alpha}}, & 0 \le y \le \theta \\ 0 & elsewhere \end{cases}$$

Where $\alpha > 0$ is a known, fixed value, but θ is unknown. Consider the estimator $\hat{\theta} = \max(Y_1, Y_2, \ldots Y_n)$.

a) Show that $\hat{\theta}$ is a biased estimator for θ
b) Find a multiple of $\hat{\theta}$ that is an unbiased estimator of θ
c) Derive MSE($\hat{\theta}$)

3. Suppose $Y_1, Y_2, \ldots Y_n$ from a random sample from a uniform distribution defined on the interval$[\theta - \frac{1}{2}, \quad \theta + \frac{1}{2}]$. Let $\hat{\theta}_1 = \frac{Y_1 + Y_2}{2}$ and $\hat{\theta}_2 = \overline{Y}$.

a) Find MSE($\hat{\theta}_1$) and MSE($\hat{\theta}_2$)
b) Which estimator, $\hat{\theta}_1$ or $\hat{\theta}_2$, is a better estimator? Why?

4. Suppose that X is a single observation from an exponential distribution with mean θ. Use the pivotal quantity $\frac{2X}{\theta}$ to derive a 90% confidence interval for θ.

5. Let X_1 and X_2 from a random sample of size 2 such that $X_i \sim Unif(0, \theta)$. The range is defined by R=max (X_1, X_2)-min (X_1, X_2). The distribution of R is $f(r) = \left(\frac{2}{\theta^2}\right)(\theta - r)$, 0<r<$\theta$.

a) Verify that $\frac{R}{\theta}$ is a pivotal quantity.
b) Construct 90% confidence interval for θ using the pivotal quantity $\frac{R}{\theta}$

6. Find a pivotal quantity based on a random sample of size n from a $N(\theta, \theta)$ population, where $\theta > 0$. Use a pivotal quantity of your choice to set up a $(1-\alpha)$ confidence interval for θ.

7. Let X be a single observation from the $Beta(\theta, 1)$ pdf, where θ is an unknown parameter.
   1) Let $U_1 = -(logX)^{-1}$. The confidence interval for θ constructed by U_1 is $\left[\frac{U_1}{2}, U_1\right]$. Evaluate the confidence coefficient.
   2) Show that $U_2 = X^{\theta}$ is a pivot.
   3) Use U_2 to set up a confidence interval having the same confidence coefficient as the interval in part 1)

8. $X_i \sim iid\ Uniform(0, \theta)$. Let $X_{(n)} = \max(X_1, X_2, \dots X_n)$.
   1) Show that $U = \frac{X_{(n)}}{\theta}$ is a pivot.
   2) Use $U = \frac{X_{(n)}}{\theta}$ as a pivotal quantity to find the shortest $(1-\alpha)$ confidence interval for θ.

9. Let $X_1, X_2, \dots X_n$ be a random sample of size n from a Poisson distribution with mean λ, where λ is an unknown parameter.
   1) Suggest an unbiased estimator for λ
   2) Suggest an unbiased estimator for $5\lambda - 2\lambda^2$

10. In a study of the relationship between birth order and college success, an investigator found that 126 in a sample of 180 college graduates were first born or only children; in a sample of 100 nongraduates of comparable age and socioeconomic background, the number of firstborn or only children was 54. Estimate the difference in the proportions of firstborn or only children for the two populations from which these samples were drawn. Give a bound for the error of estimation.