Solution — M440 — SA #1

#1 $X \sim Exp(\theta)$, $V = \frac{2X}{\theta}$

The MGF of V:

$$E_V(t) = E[e^{Vt}] = E[e^{\frac{2Xt}{\theta}}] = E[e^{X(\frac{2t}{\theta})}] = M_X(\frac{2t}{\theta})$$

since the MGF of X: $M_X(t) = \frac{1}{1-\theta t}$

Therefore $E_V(t) = M_X(\frac{2t}{\theta}) = \frac{1}{1-\theta(\frac{2t}{\theta})} = \frac{1}{1-2t}$

this is the MGF of $\chi^2(2)$

Hence, $V = \frac{2X}{\theta} \sim \chi^2(2)$

#2 X_i iid $Unif(0, \theta)$, $U = \frac{X_{(n)}}{\theta}$

since X_i iid $Unif(\theta)$, the p.d.f. of X_i is : $f_{X_i}(x) = \frac{1}{\theta}$, CDF: $F_{X_i}(x) = \frac{x}{\theta}$

The pdf of $X_{(n)}$ is: $f_{X_{(n)}}(x) = n[F(x)]^{n-1} f(x) = n(\frac{x}{\theta})^{n-1} \cdot \frac{1}{\theta} = \frac{nx^{n-1}}{\theta^n}$

using transformation method:

step1: $X_{(n)} = \theta U$

step2: θ

step3: the pdf of U: $f_U(u) = \frac{n(\theta u)^{n-1} \cdot \theta}{\theta^n} = nu^{n-1}$

$\Rightarrow f_U(u) = nu^{n-1}$ for $0 < u \leq 1$

(#3) $X_1, X_2, \dots X_5 \overset{iid}{\sim} \text{Exp}(20)$

(a) $U = \sum_{i=1}^{5} X_i$

The MGF of X_i is: $M_{X_i}(t) = \frac{1}{1-20t}$

The MGF of U is: $M_U(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_5}(t) = \left(\frac{1}{1-20t}\right)^5$

which is the MGF of Gamma(5, 20)

$\Rightarrow U \sim \text{Gamma}(5, 20)$

$\Rightarrow f_U(u) = \frac{1}{\Gamma(5)\, 20^5} u^4 e^{-u/20}$

(b) $E(U) = \alpha\beta = 5 \times 20 = 100$

$Var(U) = \alpha\beta^2 = 5 \times 20^2 = 2000$

(c) since $X_i \overset{iid}{\sim} \text{Exp}(20)$, for $i = 1, 2, \dots 5$,

And then $\mu_{X_i} = 20$, $\sigma_{X_i}^2 = 20^2 = 400$

So, $E(\bar{X}) = \mu_{X_i} = 20$

$Var(\bar{X}) = \frac{\sigma_{X_i}^2}{n} = \frac{400}{5} = 80$

(d) From problem #1, notice that if $X_i \overset{iid}{\sim} \text{Exp}(20)$

then $V = \frac{2X_i}{\theta} \sim \chi^2(2)$, that's $V_i = \frac{2X_i}{20} = \frac{X_i}{10} \sim \chi^2(2)$

Therefore, $\sum_{i=1}^{5} V_i = \sum_{i=1}^{5} \left(\frac{X_i}{10}\right) \sim \chi^2(10)$

So, $P\left(\sum_{i=1}^{5} X_i > c\right) = P\left(\sum_{i=1}^{5}\left(\frac{X_i}{10}\right) > \frac{c}{10}\right) = P\left(\chi^2(10) > \frac{c}{10}\right) = 0.05$

$\frac{c}{10} = 18.307 \implies c = 183.07$

#4 $X \sim N(5,15)$ and $Y \sim \chi^2(5)$, Determine $P(X-5>2\sqrt{Y})$

<u>Solution</u>: since $X \sim N(5,15)$,

$$\frac{X-5}{\sqrt{15}} \sim N(0,1)$$

$$T=\frac{\left(\frac{X-5}{\sqrt{15}}\right)}{\sqrt{\frac{Y}{5}}} = \frac{Z}{\sqrt{\frac{\chi^2(5)}{5}}} \sim t(5)$$

That's $T=\dfrac{X-5}{\sqrt{3}\sqrt{Y}} \sim t(5)$

Hence, $P(X-5>2\sqrt{Y}) = P\left(\frac{X-5}{\sqrt{Y}}>2\right) = P\left(\frac{X-5}{\sqrt{3}\sqrt{Y}}>\frac{2}{\sqrt{3}}\right)$

$= P\left(t(5)>\frac{2}{\sqrt{3}}\right) = P(t(5)>1.1547) > 0.1$ since t-table doesn't have 1.1547

#5 population 1: $X \sim N(\mu, 6.4^2)$, with $n=64$
population 2: $Y \sim N(\mu, 7.2^2)$, with $n=64$

clearly, $\left.\begin{array}{l} \bar{X} \sim N\left(\mu, \frac{6.4^2}{64}\right) \\ \bar{Y} \sim N\left(\mu, \frac{7.2^2}{64}\right) \end{array}\right] \Rightarrow \bar{X}-\bar{Y} \sim N\left(0, \frac{6.4^2+7.2^2}{64}\right)$

compute: $P(|\bar{X}-\bar{Y}|>0.6) = 1-P(|\bar{X}-\bar{Y}|<0.6) = 1-P(-0.6<\bar{X}-\bar{Y}<0.6)$

$$=1-P\left(\frac{-0.6}{\sqrt{\frac{6.4^2+7.2^2}{64}}}<Z<\frac{0.6}{\sqrt{\frac{6.4^2+7.2^2}{64}}}\right) = 1-P(-0.5<Z<0.5)$$

$=0.617$

#6 $X_1, X_2, \ldots X_{10} \sim N(0, \sigma^2)$, $P(-c \le \frac{S}{\bar{X}} < c) = 0.95$

(a). Find distribution of $\frac{10\bar{X}^2}{S^2}$

Solution: $\bar{X} \sim N(0, \frac{\sigma^2}{10})$

So, $\frac{\bar{X}}{\sigma/\sqrt{10}} = \frac{\sqrt{10}\,\bar{X}}{\sigma} \sim N(0,1)$

So, $\left(\frac{\bar{X}}{\sigma/\sqrt{10}}\right)^2 = \frac{10\bar{X}^2}{\sigma^2} \sim \chi^2(1)$ (*)

Also, Note that $\boxed{\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)}$

Here, $\frac{9S^2}{\sigma^2} \sim \chi^2(n-1)$ (**)

Therefore, base on (*) and (**)

we have, $$\frac{10\bar{X}^2}{S^2} = \frac{\left(\frac{10\bar{X}^2}{\sigma^2}\right)/1}{\left(\frac{9S^2}{\sigma^2}\right)/9} = \frac{\chi^2(1)/1}{\chi^2(9)/9} \sim F(1,9)$$

Method 2: for part (a):

since $\frac{\bar{X}-0}{S/\sqrt{n}} \sim t(n-1)$

that's $\frac{\bar{X}}{S/\sqrt{10}} = \frac{\sqrt{10}\,\bar{X}}{S} \sim t(9)$

Note that, if $Y \sim t(n)$ then $Y^2 \sim F(1,n)$

Here, $\left(\frac{\bar{X}}{S/\sqrt{10}}\right)^2 = \frac{10\bar{X}^2}{S^2} \sim F(1,9)$

(b) Find the distribution of $\frac{S^2}{10\bar{X}^2}$

Solution: From part (a), we know that $\frac{10\bar{X}^2}{S^2} \sim F(1,9)$

Therefore, $\frac{S^2}{10\bar{X}^2} \sim F(9,1)$

(c) Find c, such that $P(-c \le \frac{S}{\bar{X}} \le c) = 0.95$

Solution: $P(-c \le \frac{S}{\bar{X}} \le c) = P(|\frac{S}{\bar{X}}| \le c) = P(\frac{S^2}{10\bar{X}^2} \le \frac{c^2}{10}) = P(F(9,1) \le \frac{c^2}{10})$

$= 0.95$

$\Rightarrow c = 49.04$

#7 $x_i \overset{iid}{\sim} (\mu, 10^2)$, $n=100$

Find $P(|\bar{x}-\mu|<1.)$

Since $n=100$, we can use C.L.T.

$$\Rightarrow \bar{X} \sim N(\mu, \frac{100}{100}) \Rightarrow \bar{X} \sim N(\mu, 1)$$

$$\Rightarrow \frac{\bar{x}-\mu}{1} \sim N(0,1)$$

So, $P(|\bar{x}-\mu|<1) = P(|Z|<1) = 0.6826$

#8 $X_1, X_2, \dots X_{40} \overset{iid}{\sim} f(x_i) = 3x_i^2$ for $0 \le x \le 1$

Clearly, $x_i \sim Beta(3,1)$

So, $E(X_i) = \frac{\alpha}{\alpha+\beta} = \frac{3}{3+1} = \frac{3}{4}$

$$Var(X_i) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{3}{4^2 \cdot 5} = \frac{3}{80}$$

Since $n=40>30$, then apply C.L.T.

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1), \text{ that's } \frac{\bar{x}-\frac{3}{4}}{\sqrt{\frac{3}{80}}/\sqrt{40}} \sim N(0,1).$$

$$\Rightarrow P(\bar{x}>0.7) = P\left(Z > \frac{0.7-\frac{3}{4}}{\sqrt{3/80/40}}\right) = P(Z>-1.63) = 0.9484$$

It's very likely that average proportion is greater than 0.7.

#9. $X_i \overset{iid}{\sim} N(i, i^2)$. for $i = 1, 2, 3$

(a) $X_1 \sim N(1, 1^2)$, $X_2 \sim N(2, 2^2)$, $X_3 \sim N(3, 3^2)$

$$Z_1 = \frac{X_1 - 1}{1}, \quad Z_2 = \frac{X_2 - 2}{2}, \quad Z_3 = \frac{X_3 - 3}{3}$$

$$Z_1^2 + Z_2^2 + Z_3^2 = \underbrace{(X_1 - 1)^2}_{\sim \chi^2(1)} + \underbrace{\left(\frac{X_2 - 2}{2}\right)^2}_{\sim \chi^2(1)} + \underbrace{\left(\frac{X_3 - 3}{3}\right)^2}_{\sim \chi^2(1)} \sim \chi^2(3).$$

(b). $\frac{X_1 - 1}{1} \sim N(0, 1)$

$$\left(\frac{X_2 - 2}{2}\right)^2 + \left(\frac{X_3 - 3}{3}\right)^2 \sim \chi^2(2)$$

$$T = \frac{(X_1 - 1)/1}{\sqrt{\left[\left(\frac{X_2 - 2}{2}\right)^2 + \left(\frac{X_3 - 3}{3}\right)^2\right]/2}} \sim t(2)$$

(c) square the R.V. T in part (b). to obtain $F(1, 2)$.

#10 CLT

$\overline{X}_1 \sim N(\mu, \frac{\sigma^2}{n})$, $\overline{X}_2 \sim N(\mu, \frac{\sigma^2}{n})$. $\overline{X}_1 \perp \overline{X}_2$.

$\overline{X}_1 - \overline{X}_2 \sim N(0, \frac{2\sigma^2}{n})$.

Thus, $P(|\overline{X}_1 - \overline{X}_2| < \frac{\sigma}{5}) = P\left(\left|\frac{\overline{X}_1 - \overline{X}_2}{\sigma\sqrt{\frac{2}{n}}}\right| < \frac{\sigma/5}{\sigma \cdot \sqrt{\frac{2}{n}}}\right)$

$= P\left(-\frac{1}{5}\sqrt{\frac{n}{2}} < Z < \frac{1}{5} \cdot \sqrt{\frac{n}{2}}\right) = 0.99$

$n = 50(2.576)^2 \approx 332$.