1. [12 *pts*] Let $X_1, \ldots, X_n$ denote a random sample from the population with the probability density function given by

$$f(x|\theta) = \frac{3x^2}{\theta^3}, \qquad 0 \le x \le \theta,$$

where θ is unknown. Let $U = \frac{X_{(n)}}{\theta}$, where $X_{(n)} = max(X_1, \ldots, X_n)$.

   (a) [5 *pts*] Show that U is a pivotal quantity.

   (b) [5 *pts*] Find the $(1-\alpha)\%$ lower confidence bound for θ.

   (c) [2 *pts*] Supposed that we take a sample of size $n = 10$ from such population. Find a 90% lower confidence bound for θ assuming we observed that $X_{(n)} = 25$. .

2. [10 *pts*] Let $X_1, \ldots, X_n$ be a random sample from the density function

$$f(x|\alpha, \beta) = \alpha\beta^\alpha x^{-(\alpha+1)}, \quad x \ge \beta$$

where α and β are both unknown parameters.

   (a) [5 *pts*] Find the joint sufficient statistics for α and β.

   (b) [5 *pts*] Find the MLE of α and β, respectively. Use the graph to support your answer, if needed.

3. [8 *pts*] A retail dealer sells three brands of automobiles. For brand A, her profit per sale, $X \sim N(\mu_1, \sigma_1^2)$; for brand B, her profit per sale, $Y \sim N(\mu_2, \sigma_2^2)$; for brand C, her profit per sale, $W \sim N(\mu_3, \sigma_3^2)$. For the year, two-fifths of the dealer's sales are of brand A, one-fifth of brand B, and the remaining two-fifths of brand C. If you are given data on profits for n_1, n_2 and n_3 sales of brands A, B, and C, respectively, the quantity

$$U = 0.4\bar{X} + 0.2\bar{Y} + 0.4\bar{W}$$

will approximate to the true average profit per sale for the year, where $\bar{X}$,$\bar{Y}$, and $\bar{W}$ represent the sample mean for the sales of brands A, B, and C, respectively. Assume X, Y, and W are independent. What distribution does U follow? Show the proof. Find the mean and variance of U.

4. [19 *pts*] Let $X_1, \ldots, X_n$ denote a random sample from the probability density function given by

$$f(x|\theta) = \theta x^{\theta-1}, \qquad 0 < x < 1$$

where $\theta > 0$ is unknown. Let $W_i = -\ln X_i$,

   (a) [5 *pts*] Show that $U = \sum_{i=1}^n W_i$ is a sufficient statistic for θ.

   (b) [6 *pts*] Let $V_i = -2\theta \ln X_i$, show that $2\theta U = \sum_{i=1}^n V_i$ has a χ^2 distribution with the degree of freedom $2n$.

   (c) [4 *pts*] Show that

$$E[\frac{1}{2\theta U}] = \frac{1}{2(n-1)}$$

   Hint for (c): if $Y \sim Gamma(\alpha, \beta)$, then $E[Y^a] = \frac{\beta^a \tau(\alpha+a)}{\tau(\alpha)}$ for any positive or negative values of a.

   (d) [4 *pts*] Find the MVUE for θ.

5. [12 *pts*] Researchers have shown that cigarette smoking has a deleterious effect on lung function. In their study of the effect of cigarette smoking on the carbon monoxide diffusing capacity (DL) of the lung, Ronald Knudson, Walter Kaltenborn, and Benjamin Burrows (American Review of Respiratory Diseases 140, 1989, pp.645-51) found that current smokers had DL readings significantly lower than either ex-smokers or nonsmokers. The carbon monoxide diffusing capacity for a random sample of current smokers was as follows:

$$102 \quad 92 \quad 106 \quad 70 \quad 95$$

These measurement have a mean of 93 and a standard deviation of 14.

(a) [5 *pts*] Do these data indicate that the mean DL reading for current smokers is lower than 100, the average DL reading for nonsmokers? Use $\alpha = .05$. Your answer must include the hypothesis, the value of the test statistic, the p-value, and your decision.

(b) [5 *pts*] Find a 95% upper confidence bound for true mean DL reading for current smokers. Interpret the confidence interval.

(c) [2 *pts*] How does the conclusion that you reached in part (a) compare with your conclusion in part (b).

6. [14 *pts*] A sample of size 1 is taken from the probability density function

$$f_X(x) = (\theta + 1)x^{\theta}, \qquad 0 \le x \le 1.$$

We want to test

$$H_0 : \theta = 1 \qquad H_a : \theta > 1.$$

Let X be the test statistic, and define the rejection region $RR = \{X \ge 0.9\}$.

(a) [5 *pts*] Find the significance level of the test α.

(b) [5 *pts*] Find an expression for β as a function of θ.

(c) [4 *pts*] Sketch a graph of the power function.

7. [13 *pts*] Let $X_1, \ldots, X_{10}$ be a random sample from a $Bernoulli(p)$ with the probability mass function

$$p(x_i|p) = p^{x_i}(1-p)^{1-x_i}$$

where p is unknown.

(a) [8 *pts*] Find the most powerful test of size $\alpha = 0.0547$ of the hypotheses

$$H_0 : p = \frac{1}{2} \qquad H_a : p = \frac{1}{4}.$$

And calculate the power of the test.

(b) [5 *pts*] For testing

$$H_0 : p \le \frac{1}{2} \qquad H_a : p > \frac{1}{2}.$$

Find the level of the test that rejects H_0 if $\sum_{i=1}^{10} X_i \ge 6$.

8. [12 *pts*] Suppose $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ with unknown mean μ and unknown variance σ^2. Construct a likelihood Ratio Test (LRT) of size $\alpha = 0.05$ of

$$H_0 : \mu \ge \mu_0 \qquad H_a : \mu < \mu_0.$$