Degenerate electron gas

$\downarrow$

quantum statistics [Fermi-Dirac] must be applied

$$k_B T \ll E_F = \frac{p_F^2}{2m} = (3\pi^2)^{2/3} \frac{\hbar^2}{m} n^{2/3}$$

Chemical potential $\mu = E_F -$ corrections $O\left[\left(\frac{k_B T}{E_F}\right)^2\right]$

Occupation numbers

large

$$n = \frac{1}{e^{\frac{\varepsilon-\mu}{k_B T}}+1} \approx 1 - e^{-\frac{\mu-\varepsilon}{k_B T}} \approx 1 \qquad \forall \varepsilon$$

$$+\frac{\mu-\varepsilon}{k_B T} \gg 1$$

Plasma (neutral): electrons of density n
+ ions $\oplus Ze$ $\quad n_i = \frac{1}{Z} n$

e-ion interaction $\frac{Ze^2}{a} \ll E_F$ kinetic energy per one electron

$$a \sim n_i^{-1/3} = (Z/n)^{1/3}$$

mean distance between ions

$$Ze^2 \cdot \left(\frac{n}{Z}\right)^{1/3} \ll \frac{\hbar^2}{m} n^{2/3}$$

$$Z^{2/3} \ll \frac{\hbar^2}{me^2} n^{1/3}$$

Bohr radius

$$n \gg Z^2 \cdot \left(\frac{\hbar^2}{me^2}\right)^{-3} = Z^2 a_0^{-3}$$

$$\boxed{n \cdot a_0^3 \gg Z^2}$$

The higher density of degenerate electron gas, the more ideal it is:

$$K \gg U_{e\text{-ion}}, U_{e\text{-}e}$$

Thermal properties of degenerate electron gas

$k_B T \ll E_F$

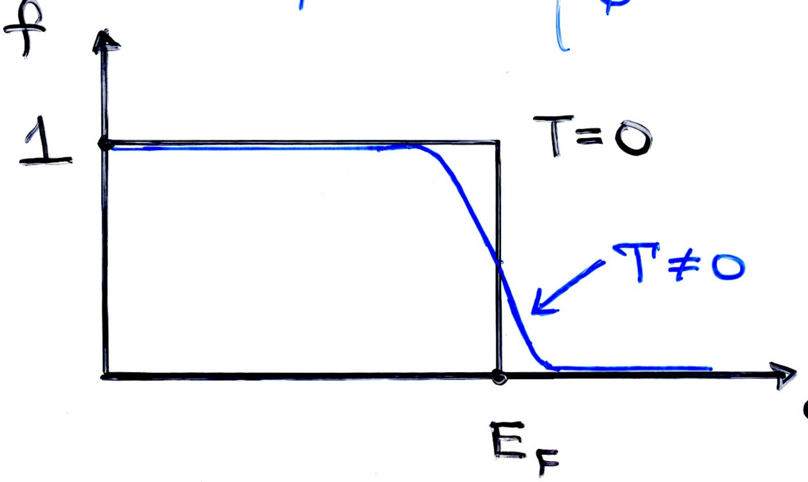
ground state $T=0$ + corrections $\sim \left(\frac{k_B T}{E_F}\right)$ some power

the only source of C_V ...

$$\boxed{f(\varepsilon) = \frac{1}{\exp([\varepsilon-\mu]/k_B T)+1}}$$

$T \to 0$

$$f(\varepsilon) = \Theta(\mu-\varepsilon) = \begin{cases} 1 & \varepsilon < \mu \\ 0 & \varepsilon > \mu \end{cases}$$



$, \quad \frac{k_B T}{E_F} \ll 1$

$\mu(T=0) = E_F$

Condition for finding μ:

$$n = \int_{-\infty}^{\infty} d\varepsilon\, f(\varepsilon)\cdot g(\varepsilon)$$

Density of states

$$n = 2\cdot\int \frac{d^3k}{(2\pi)^3} f(\varepsilon_k)$$

electron density

$$d^3k = 4\pi k^2 dk = 4\pi\cdot\left(\frac{2m\varepsilon}{\hbar}\right)\cdot\frac{1}{2}\left(\frac{2m}{\hbar}\right)^{1/2}\varepsilon^{-1/2}d\varepsilon$$

$$\frac{\hbar^2 k^2}{2m} = \varepsilon \qquad k = \left(\frac{2m\varepsilon}{\hbar}\right)^{1/2} \qquad dk = \frac{1}{2}\left(\frac{2m}{\hbar}\right)^{1/2}\varepsilon^{-1/2}d\varepsilon$$

$$\boxed{g(\varepsilon) = \frac{(2m)^{3/2}}{2\pi^2\hbar^3}\sqrt{\varepsilon}}$$

DOS per unit volume $V=1$ for both $\uparrow\downarrow$ directions of spin

DOS

$$g(\varepsilon) = \begin{cases} \frac{3}{2} \cdot \frac{n}{E_F} \left(\frac{\varepsilon}{E_F}\right)^{1/2}, & \varepsilon > 0 \\ 0, & \varepsilon < 0 \end{cases}$$

$$E_F = \frac{\hbar}{2m} \cdot (3\pi^2 n)^{2/3}$$ used

We can calculate now various thermodynamic quantities:

Internal energy density

$$u = \int_{-\infty}^{\infty} d\varepsilon\, f(\varepsilon)\, g(\varepsilon) \cdot \varepsilon \qquad \text{etc.}$$

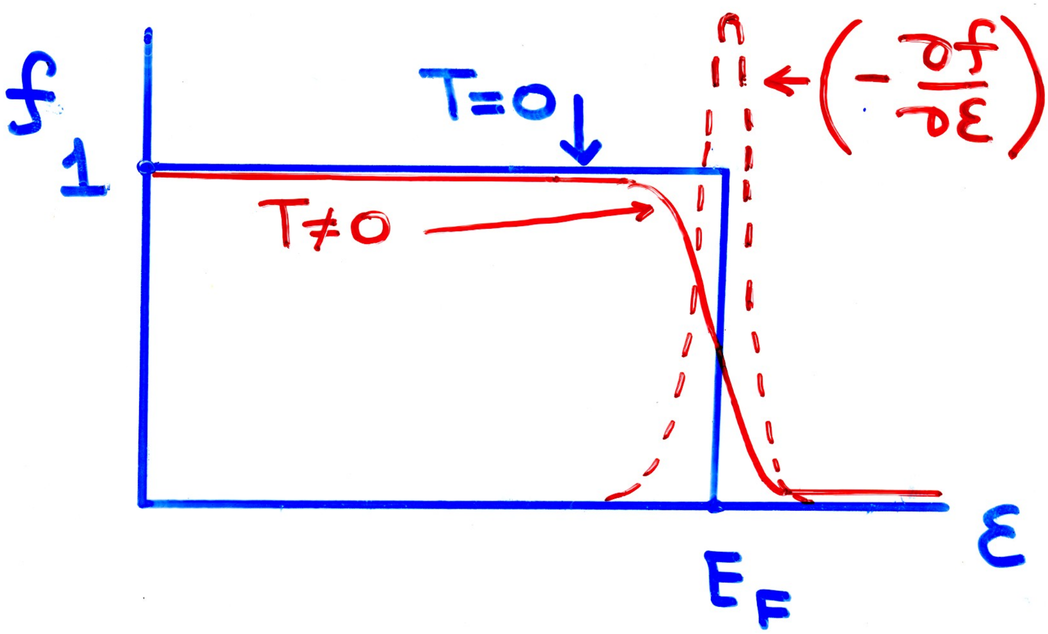
For metals $T_{room} \ll E_F \sim (2-25) \times \underbrace{12 \times 10^{3}}_{1eV} K$

Low-temperature expansion Sommerfeld

$$\int_{-\infty}^{\infty} d\varepsilon \cdot f(\varepsilon) \cdot H(\varepsilon) = \int_{-\infty}^{\mu} d\varepsilon \cdot H(\varepsilon) + \sum_{n=1}^{\infty} (k_B T)^{2n} \cdot a_n \left. \frac{d^{2n-1} H(\varepsilon)}{d\varepsilon^{2n-1}} \right|_{\varepsilon = \mu}$$

dimensionless constants of order unity

$$a_1 = \frac{\pi^2}{6} \qquad a_2 = \frac{7\pi^4}{360}$$



$$f(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon-\mu}{k_BT}\right)+1} \qquad x = \frac{\varepsilon-\mu}{k_BT}$$

$$-\frac{\partial f}{\partial \varepsilon} = \frac{1}{k_BT}\cdot\frac{e^x}{\left[e^x+1\right]^2} = \frac{1}{k_BT}\cdot\frac{1}{4\cosh^2\frac{x}{2}}$$

even

$$\int_{-\infty}^{\infty} d\varepsilon \left(-\frac{\partial f}{\partial \varepsilon}\right) K(\varepsilon) = K(\mu) + \text{corrections}$$

$k_BT \ll E_F$

behaves like $\delta(\varepsilon-\mu)$ + corrections

$$T=0: \; f(\varepsilon) = \Theta(\mu-\varepsilon) \qquad -\frac{\partial f}{\partial \varepsilon} = \delta(\varepsilon-\mu)$$

$$\int_{-\infty}^{\infty} d\varepsilon\, f(\varepsilon) H(\varepsilon) = \int_{-\infty}^{\mu} d\varepsilon\, H(\varepsilon) + \frac{\pi^2}{6} (k_B T)^2 H'(\mu) + \dots$$

If $H(\varepsilon)$ is a smooth function with variations on the energy scale $\sim \mu$

$H' \sim \frac{H}{\mu}$

$\Rightarrow$ expansion in $\frac{k_B T}{E_F} \ll 1$

Examples: T-corrections to

① internal energy

② chemical potential

① $u = \int_{-\infty}^{\infty} d\varepsilon\, f(\varepsilon) \boxed{g(\varepsilon)\varepsilon}$

$H(\varepsilon)$

$H'(\varepsilon) = g + g' \cdot \varepsilon$

② $n = \int_{-\infty}^{\infty} d\varepsilon\, f(\varepsilon) \boxed{g(\varepsilon)}$

$H'(\varepsilon) = g'(\varepsilon)$

$$g(\varepsilon) = \frac{(2m)^{3/2}}{2\pi^2 \hbar^3} \sqrt{\varepsilon} \quad \text{DOS}$$

$$\int_0^{\mu} d\varepsilon\, H(\varepsilon) \simeq \int_0^{E_F} d\varepsilon\, H(\varepsilon) + H(E_F) \cdot (\mu - E_F)$$

Chemical potential

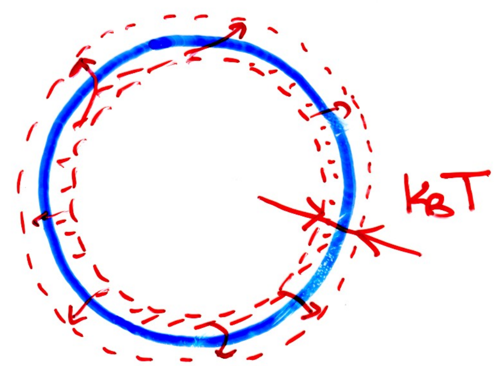
$$\mu = E_F\left[1 - \frac{\pi^2}{12}\cdot\left(\frac{k_B T}{E_F}\right)^2\right]$$

$$u = u_0 + \underbrace{\frac{\pi^2}{6}(k_B T)^2\cdot g(E_F)}$$

thermal energy density

$g(E_F) = \frac{3}{2}\cdot\frac{n}{E_F}$ DOS at the Fermi energy

correction $\delta u \sim k_B T\cdot\frac{k_B T}{E_F}$



Physical meaning:

$\Delta N \sim k_B T\cdot g(E_F)$

\# of excited electrons

$\delta u \sim k_B T\cdot\Delta N \sim (k_B T)^2\cdot g(E_F)$

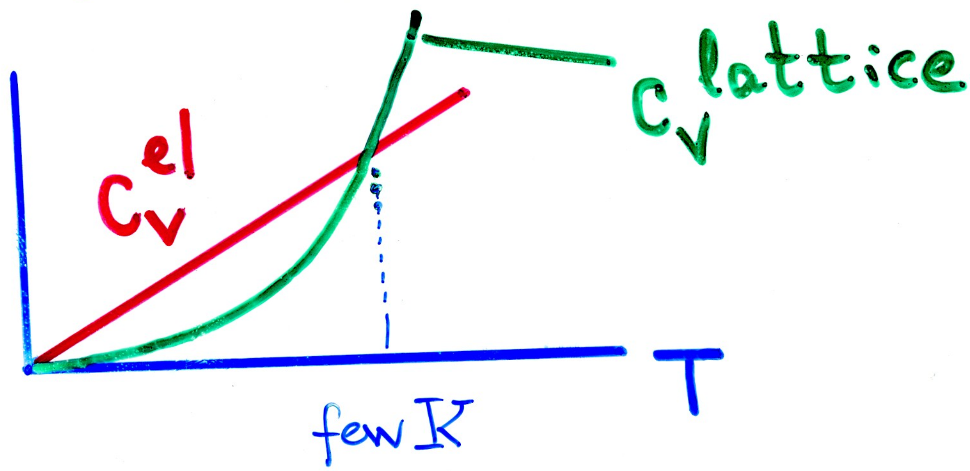
$$C_V = \left(\frac{\partial u}{\partial T}\right)_V = \frac{\pi^2}{3}k_B^2 T\cdot g(E_F)$$

$$\boxed{C_V = \frac{\pi^2}{2}\cdot\frac{k_B T}{E_F}\cdot n k_B \sim \frac{k_B T}{E_F}\cdot C_V^{class}}$$

$$\boxed{C_V^{el} \sim T}$$

$$C_V = C_V^{el} + C_V^{lattice} \quad , \quad C_V^{lattice} \sim T^3$$

$$C_V = \gamma \cdot T + A \cdot T^3$$



Specific heat effective mass m^*

$$C_V^{el} \sim k_B^2 T \cdot g(E_F) \sim n/E_F \sim \left(\frac{\hbar^2}{m r_s^2}\right)^{-1} \sim m$$

electron mass

$$C_V^{el} = \frac{m^*}{m} \cdot \gamma T$$

for free electrons of density n

$m^* \sim 1$ Alkali, Noble metals

$m^* \sim 10^{-2} - 10^{-1}$ Bi, SB

$m^* \sim 10$ Mn, Fe ??

Conduction in metals

Boltzmann distribution → Fermi-Dirac distr.

①. $$f(\vec{v}) = \frac{(m/\hbar)^3}{4\pi^3} \cdot \frac{1}{\exp\left(\left[\frac{mv^2}{2} - \mu\right]/k_B T\right) + 1}$$

②. (Quasi) classical description of electron motion

$\Delta x \cdot \Delta p \sim \hbar$ $\qquad$ $\Delta x \ll \ell \, , \, \lambda$

mean free path $\qquad$ wave length, charact. length

$$\Delta p \gg \frac{\hbar}{\ell}, \frac{\hbar}{\lambda} \ll \hbar k_F = p_F$$

$$k_F \sim r_s^{-1}$$

$$\boxed{\ell, \lambda \gg r_s}$$

Mean free path $\qquad$ $\ell = v_F \cdot \tau \sim 10^8 \frac{cm}{sec} \times 10^{-14}\, sec$

$$\ell \sim 10^{-6}\, cm = 100 \mathring{A}$$

large compared to lattice spacing !

Thermal conductivity

$$K = \frac{1}{3} v^2 \tau C_V$$

$$v^2 \rightarrow v_F^2 \qquad C_V \rightarrow \sim \frac{k_B T}{E_F} \cdot C_V^{class} = \frac{\frac{1}{3} \cdot \frac{m v_{class}^2}{2}}{\frac{m v_F^2}{2}} \cdot C_V^{class}$$

quantum contributions compensate

Electrical conductivity

$$\sigma = \frac{n e^2 \tau}{m}$$

does not change <u>if</u> τ does not depend on energy $\Leftrightarrow$ on velocity distribution

Lorenz number

Somm. $\frac{K}{\sigma T} = \boxed{\frac{\pi^2}{3}} \left(\frac{k_B}{e}\right)^2 = 2.4 \times 10^{-8} \, \frac{\text{watt} \cdot \text{ohm}}{K^2}$

$\simeq 3.3$

excellent agreement with exp.

Drude $\frac{K}{\sigma T} = \frac{3}{2} \cdot \left(\frac{k_B}{e}\right)^2 = 1.1 \times 10^{-8}$ — " —

23.10

Thermopower Q $\quad \vec{E} = Q\,\vec{\nabla}T$

$$Q = -\frac{c_V}{3ne}$$

$$\boxed{Q = -\frac{\pi^2}{6}\frac{k_B}{e}\cdot\frac{k_BT}{E_F}}$$

Somm. $c_V = \frac{\pi^2}{2}\left(\frac{k_BT}{E_F}\right)\cdot n k_B$

Drude $c_V^{class} = \frac{3}{2} n k_B$: $\left[\frac{k_BT}{E_F}\right]^{-1} \sim 10^2$ overestimation

Solids: Is classical description of electron kinetics worthless? (stat.)

No! : Non-degenerate electron gas
$\quad \shortparallel$ classical

low densities of electrons

$$\frac{\hbar^2}{m r_s^2} \sim \frac{\hbar^2}{m}\cdot n^{2/3} \ll k_BT$$

$$r_s \gg \left(\frac{\hbar^2}{m k_B T}\right)^{1/2}$$

Semiconductors