Math- 440, Lab 5:
Tuesday May 10, 2005

(1) Suppose that $X_1, ..., X_n$ form a random sample from a normal distribution for which both the mean $\mu$ and the variance $\sigma^2$ are unknown. Let $\hat{\sigma}^2$ and $S^2$ be the two estimators of $\sigma^2$, which are defined as follows:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})$$

and

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})$$

show that $\hat{\sigma}^2$ has a smaller MSE than $S^2$.

(2) Let $X_1, ..., X_n$ be iid Bernoulli$(p)$. Calculate:

(a) The MSE of $\overline{X}$.

(b) Let $Y = \sum_{i=1}^{n}$. Calculate the MSE of the Bayes estimator namely the MSE of $\hat{p}_b$ as shown as follows:

$$\hat{p}_b = \frac{Y + \alpha}{\alpha + \beta + n}$$

where $\alpha$ and $\beta$ are parameters of a Beta prior distribution.

(c) It is easy to prove that by choosing $\alpha = \beta = \sqrt{n/4}$, the MSE of $\hat{p}_b$ becomes a constant. Using these values, compare the MSE of $\overline{X}$ with the MSE of $\hat{p}_b$ for $n = 4$ and $n = 400$.

(3) Let $X_1, ..., X_n$ be a random sample from a Normal$(\mu, 25)$ distribution. Consider $H_0 : \mu \leq 17$. Obtain the power function for this test when $n = 25$. 
(4) Assume that $X_1, \ldots, X_n$ is iid from a $\text{Normal}(\mu, \sigma^2)$. Now consider the hypothesis testing problem for the variance of the model. That is, we want to test: $H_0: \sigma = \sigma_0$ and one of the following alternatives:

- $H_a: \sigma > \sigma_0$
- $H_a: \sigma < \sigma_0$
- $H_a: \sigma \neq \sigma_0$.

(a) Suggest a hypothesis testing procedure for this problem. That is, determine a proper test statistic, sketch the critical region and advise as how to calculate the $p$-value.

(b) Address the same issues for testing $H_0: \sigma_1^2 = \sigma_2^2$, where $\sigma_1^2$ and $\sigma_2^2$ are unknown variances of two normal distributions.

(5) Notice that the power of the Wald test can be presented as:

$$1 - \Phi\left(\frac{\theta_0 - \theta^*}{\hat{\sigma}e} + z_{\alpha/2}\right) + \Phi\left(\frac{\theta_0 - \theta^*}{\hat{\sigma}e} - z_{\alpha/2}\right)$$

In class, we claimed that the power of this test increases when two things happen: 1) $\theta_0$ gets farther away from $\theta^*$, and 2) $\hat{\sigma}e \to 0$ which mainly happens when the sample size gets larger. Explain why this claim is valid?

(6) Remember that the definition of the $p-value$ is as follows:

$$p-value = \inf\{\alpha; T(X) \in C_\alpha\}$$ for $C_\alpha$, the critical region of a size $\alpha$ test. In this sense, $p-value$ is the smallest level at which we reject $H_0$. Show that the $p-value$ can be written as follows:

$$p-value = \sup_{\theta \in \Omega_0} Pr(T(X) \geq T(x)).$$