(1) (Freund, 1992) Find the mean of the posterior distribution as an estimate of the true probability of a success, if 42 successes are obtained in 120 Bernoulli trials and the prior distribution of $\theta$ is a Beta with parameters $\alpha = \beta = 40$.

(2) (Freund, 1992) A distributor of soft-drink vending machines feels that in a supermarket one of his machines will sell on the average $\mu = 738$ drinks per week. Of course, the mean will vary somewhat from market to market, and the distributor feels that this variation is measured by the standard deviation $\nu = 13.4$. So far, as a machine placed in a particular market is concerned, the number of drinks sold will vary from week to week, and this variation is measured by the standard deviation $\sigma = 42.5$. If one of the distributor’s machines put into a new supermarket averaged $\bar{x}_n = 692$ during first 10 weeks, what is the probability that the posterior average of the drinks sold is between 700 and 720?

(3) Suppose that we have an iid sample from a Uniform$(0, \theta)$ distribution with $\theta > 0$ unknown. If the prior distribution of $\theta$ is Gamma$(\alpha, \beta)$, then obtain the form of the posterior density of $\theta$.

(4) Let $X_1, ..., X_n$ be iid Bernoulli. Then, $Y = \sum X_i$ follows a binomial$(n, p)$ distribution.

(a) Obtain the joint distribution of $Y$ and $p$.

(b) Obtain the marginal distribution of $Y$.

(c) Obtain the Posterior distribution of $p|y$.

(d) Show that the posterior mean can be written as a linear combination of the prior mean and the sample proportion.
(5) Let $X_1, \ldots, X_n$ be an iid sample from a Beta($\alpha, 1$) distribution, where $\alpha > 0$ is unknown.

(a) Determine the likelihood function.

(b) Determine the maximum likelihood estimator of $\alpha$ (assume that $\Gamma(\alpha)$ is a differentiable function of $\alpha$.)

(6) Let $X_1, \ldots, X_n$ be an iid sample from a Gamma($\alpha_0, \theta$) distribution where $\alpha_0 > 0$ and $\theta \in (0, \infty)$ is unknown, then determine the MLE of $\theta$.

(7) Let $X_1, \ldots, X_n$ be an iid sample from a Pareto($\alpha$) distribution (see below), where $\alpha > 0$ is unknown, then determine the MLE of $\alpha$.

We say that the random variable $X$ follows a Pareto($\alpha$) distribution if its pdf has the following form:

$$f(x|\alpha) = \alpha(1 + x)^{-\alpha - 1}$$

for $0 < x < \infty$, and $\alpha > 0$. 
