Math- 440, Lab 2: 
Tuesday April 12, 2005

(1) Suppose that we have a biased coin, where the probability of getting a head on a single toss is \( \theta = 0.6 \), and we will toss the coin \( n = 1000 \) time, and we wish to compute the probability of getting at least 550 heads and no more than 625 heads. We also want to calculate the probability of getting at least 323 tails.

(2) Suppose \( \Pr(X_n = i) = \frac{n+i}{3n+6} \), for \( i = 1, 2, 3 \). Suppose also that \( \Pr(X = i) = 1/3 \), for \( i = 1, 2, 3 \). Prove that \( X_n \) converges in distribution to \( X \).

(3) The lifetime of a TV tube (in years) is an exponential random variable with mean 10. What is the probability that the average lifetime of a random sample of 36 tubes is at least 10.5?

(4) The time it takes for a student to finish an aptitude test (in hours) has the following probability density function:

\[
f(x) = \begin{cases} 
6(x - 1)(2 - x) & 1 < x < 2 \\
0 & \text{otherwise}
\end{cases}
\]

Approximate the probability that the average length of time it takes for a random sample of 15 students to complete the test is less than 1 hour and 25 minutes.

(5) If 20 random numbers are selected independently from the interval \((0,1)\), what is the approximate probability that the sum of these numbers is at least eight?

(6) Let \( Y_1, ..., Y_n \) be independent random variables each normally distributed with \( E(Y_i) = \mu_i \) and \( Var(Y_i) = \sigma_i^2 \), for \( i = 1, ..., n \). Define \( U = \sum_{i=1}^{n} a_i Y_i \) where \( a_i \) are real constants. Prove that \( U \sim N\left( \sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2 \right) \).

(7) Let \( Z_1, ..., Z_n \) be independent random variables each normally distributed with \( E(Y_i) = 0 \) and \( Var(Y_i) = 1 \), for \( i = 1, ..., n \). Prove that \( \sum_{i=1}^{n} Z_i^2 \) follows a \( \chi^2 \) distribution with \( n \) degrees of freedom.
(8) (Bonus) **Delta Method.** Let \( X_n \) be a sequence of random variables such that:

\[
\sqrt{n}(X_n - \theta) \to N(0, \sigma^2), \text{ in distribution.}
\]

Suppose that \( g(x) \) is a differentiable function at \( \theta \) and \( g'(\theta) \neq 0 \). Show that:

\[
\sqrt{n}(g(X_n) - g(\theta)) \to N(0, \sigma^2(g'(\theta))^2), \text{ in distribution.}
\]