Extra Topics on The Bayesian Normal Model

(A1) Normal Likelihood, Normal Prior, Mean is unknown, Variance is known.
This is theorem 6.3.3. Note that the model is represented as:

Likelihood. \( f_n(x|\theta) \propto \exp\left[ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \theta)^2 \right] \).

Prior. \( g(\theta) \propto \exp\left[ -\frac{1}{2\nu^2} (\theta - \mu)^2 \right] \).

Posterior. Is normally distributed with:
\[ E(\theta|x) = \sigma^2 \frac{\mu + \nu \bar{x}}{\sigma^2 + \nu} \]
and
\[ \text{Var}(\theta|x) = \frac{\sigma^2 \nu^2}{\sigma^2 + \nu} \].

(A2) Normal Likelihood, Normal Prior, Mean is known, Variance is unknown.
Let the known mean be presented by \( \mu_0 \). We have:

Likelihood \( f_n(x|\sigma) \propto (\sigma^2)^{-\frac{n}{2}} \exp\left[ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_0)^2 \right] \).

Prior We consider an Inverse Gamma prior for the unknown variance \( \sigma^2 \). The idea is very simple:
If the random variable \( X \) follows a Gamma distribution, then \( 1/X \) is distributed inverse Gamma.
The pdf of the prior can be represented as following:
\[ \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-(\alpha+1)} \exp(-\beta/\sigma^2), \]
where, \( \sigma^2 > 0, \alpha > 0, \) and \( \beta > 0. \)

Posterior. After some simple math (multiplying the prior and the likelihood), we have the posterior distribution of \( \sigma^2 \) also follows the Inverse Gamma distribution with two parameters \( \alpha' = \alpha + \frac{n}{2} \) and \( \beta' = \beta + 1/2(\sum_{i=1}^{n} (x_i - \mu_0)) \).

(A2) Normal Likelihood, Normal Prior, Mean is unknown, Variance is unknown.
Quite interestingly, if we put priors like:
\( P(\mu) \propto c, \) and \( p(\sigma) \propto \sigma^{-1} \) for the mean and the standard deviation respectively, then the marginal posterior distributions of \( \mu \) and \( \sigma \) are as follows:

Posterior Distribution of \( \mu|x \). \( \mu|x \sim t(n - 1) \). Where \( t(n – 1) \) represents the \( t \)-distribution with \( n – 1 \) degrees of freedom.

Posterior Distribution of \( \sigma|x \). We have, \( \sigma|x \sim IG(n - 1, \frac{n-1}{2}s^2) \), where \( IG \) represents the inverse gamma distribution, and \( s^2 \) is simply the sample variance.