Math- 440, Exam 2:

Thursday May 26, 2005

(1) Suppose that \( X_1, ..., X_n \) form a random sample from a normal distribution for which both the mean \( \mu \) and the variance \( \sigma^2 \) are unknown. Let \( \hat{\sigma}^2 \) and \( S^2 \) be the two estimators of \( \sigma^2 \), which are defined as follows:

\[
S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2
\]

and

\[
\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2
\]

show that \( \hat{\sigma}^2 \) has a smaller MSE than \( S^2 \). (20 points)

(2) Calculate the power of any Wald test (10 points). Show that the power of a Wald test increases under two scenarios: 1- By choosing an alternative that is farther away from the hypothesized value, and 2- By letting the standard error of the test approaching to zero. (10 points)

(3) Suppose that \( X_1, ..., X_n \) form a random sample from a normal distribution for which mean \( \mu \) is unknown and the variance is 1, and it is desired to test the following hypotheses for \( \mu_0 = 0 \):

\[
H_0 : \mu = 0 \\
H_a : \mu \neq 0.
\]

Consider a test procedure such that the hypothesis \( H_0 \) is rejected if either \( \bar{X} \leq c_1 \) or \( \bar{X} \geq c_2 \), and let \( \beta(\mu) \) determine the power function of the test.

(a) Determine the values of the constants \( c_1 \) and \( c_2 \) such that \( \alpha = 0.05 \) and the function \( \beta(\theta) \) is symmetric with respect to \( \mu = 0 \). (10 points)

(b) Calculate the power of this test with \( n = 25 \) and \( \mu^* = 1/4, \mu^* = 0.5, \mu^* = 1 \). (10 points)
(4) A vice president in charge of sales for a large company claims that sales people are averaging no more than 15 sales contacts per week. He would like to increase the figure. As a check on his claim, $n = 36$ salespeople are selected at random, and the number of contacts made by each is recorded for a single randomly selected week. The mean of the 36 measurements was 17 and the population variance ($\sigma^2$) was approximated as 9. Let $\alpha = 0.05$.

(a) Calculate the p-value to verify the claim of the vice president through the following hypotheses: (10 points)

$$H_0 : \mu \leq 15$$
$$H_a : \mu > 15$$

(b) Find the power of this test when $\mu^* = 16$. That is, find the power of the test when there is an increase of one call in the mean number of customer calls per week. (10 points)

**Hint:** $pvalue = \sup_{\theta \in \Omega_0} Pr(T(X) \geq T(x))$.

(5) In the June 1986 issue of *Consumer Reports*, some data on the calorie content of beef hot dogs is given. Here are the numbers of calories in 20 different hot dog brands:

186, 181, 176, 149, 184, 190, 158, 139, 175, 148, 152, 111, 141, 153, 190, 157, 131, 149, 135, 132.

Assume that these numbers are observed values from a random sample of twenty independent normally distributed random variables with mean $\mu$ and variance $\sigma^2$, both unknown.

(a) Find a 90 percent confidence interval for the mean number of calories $\mu$. (5 points)

(b) Find a 90 percent confidence interval for the standard deviation of calories $\sigma$. (5 points)

(c) At $\alpha = 0.10$, test the hypotheses:

$$H_0 : \mu = 150$$
$$H_a : \mu \neq 150$$

(5 points)

(d) At $\alpha = 0.10$, test the hypotheses:

$$H_0 : \sigma^2 = 15$$
$$H_a : \sigma^2 \neq 15$$

(5 points)