A MACKEY-ARENS THEOREM FOR TOPOLOGICAL ABELIAN GROUPS

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Abstract. If \( G \) is an Abelian group with a collection \( X \) of homomorphisms into the usual torus \( T \) such that (a) \( X \) is a group under pointwise operation, and (b) the weakest topology on \( G \) that makes the elements of \( X \) continuous is Hausdorff. We give necessary and sufficient conditions for the existence of the finest locally quasi-convex group topology on \( G \), having exactly \( X \) as the group of continuous homomorphisms into \( T \). Such topology is the topology of uniform convergence on an appropriate, sometimes proper, subfamily of weakly compact quasi-convex subsets of \( X \). These results are motivated by the usual Mackey-Arens theorem for locally convex spaces.

1. Introduction and motivation

Given an Abelian topological group \((G, t)\), with underlying group \( G \) and topology \( t \), we say that a group topology \( \lambda \) on \( G \) is compatible with \( t \) if the sets of \( t \)- and \( \lambda \)-continuous homomorphisms from \( G \) into \( T \) coincide, i.e., if \((G, t)^\sim = (G, \lambda)^\sim\). For example, it is a theorem of Comfort and Ross [8] that \( \sigma \), the weakest topology on \( G \) that makes the elements of \((G, t)^\sim\) continuous, is compatible with \((G, t)\). Following the lead from the theory of locally convex spaces (LCSs), it is natural to ask (a) if \((G, t)\) has its Mackey topology, i.e., the finest group topology \( \mu \) on \( G \) compatible with \( t \), in the sense that if \( \lambda \) is a compatible group topology with \( t \), then \( \lambda \leq \mu \), and (b) if such \( \mu \) exists, is it given as the topology of uniform convergence on a certain family of subsets of the character group? These questions have been undertaken by Chasco, Martín-Peinador and Tariebadze (hereafter Chasco et al) [5]. For example, they prove that there is a topological group with no Mackey topology, thus answering (a) negatively. After giving the reasons for studying question (a) in the realm of locally quasi-convex group topologies (defined below) – a natural generalization of local convexity in the theory of topological vector spaces (TVSs) – they give conditions for its existence, and show that for certain types of groups the Mackey topology exists. As for question (b) the contributions of [5] are significant as they look at the family \( \mathcal{C} \) of weakly compact quasicompact subsets of the character group - the exact counterpart in the theory of LCS - giving necessary and sufficient conditions for \( \mathcal{C} \) to be the collection requested in (b).

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In this paper we give a positive answer to question (b), showing that if the (locally quasi-convex) Mackey topology \( \mu \) exists, then it is given as the topology of uniform convergence on an appropriate subfamily \( \mathcal{M} \) of \( \mathcal{C} \). We also contribute to the solution of question (a) giving a condition on \( \mathcal{M} \) that is both necessary and sufficient for \( \mu \) to exist. We present at the end examples that give negative answers to questions stated in \([5]\).

2. Preliminaries

We maintain the notation so far introduced. All our groups are abelian, thus we use additive notation for the group operation, denoting the identity of the group \( G \) by \( 0_G \). We identify the torus \( T := \mathbb{R}/\mathbb{Z} \) with the group \( \left(-\frac{1}{2}, \frac{1}{2}\right] \) with operation addition modulo 1. A character of \((G, t)\) (or a \( t \)-character) is a \( t \)-continuous group homomorphism from \( G \) into \( T \). We define \((G, t)\), i.e., the character group of \((G, t)\), by

\[
(G, t) := \{ h : G \rightarrow T \mid h \text{ is a } t \text{-character} \},
\]

with group operation defined pointwise:

\[
(h_1 + h_2)(g) := h_1(g) + h_2(g) \quad \forall g \in G.
\]

We use the symbol \( X \) to denote \((G, t)\). The group \((G, t)\) is said to be \textit{maximally almost periodic} (MAP) if \( X \) separates the points of \( G \), i.e., if for every \( g \in G \setminus \{0_G\} \) there is \( h \in X \) such that \( h(g) \neq 0 \). In this note, we assume that all our groups are MAP. The symbol \( d \) denotes the discrete topology; we write \( G_d \) sometimes instead of \((G, d)\). We also take \( \hat{G} := (G, d) \).

If \( A \subseteq G \), the set \( A^* \) denotes its \textit{polar} in \( \hat{G} \), and is defined by \( A^* := \{ h \in \hat{G} : |h(g)| \leq \frac{1}{r} \forall g \in A \} \). Following \([5]\), if \( A \subseteq G \), the set \( A^o \) denotes its \textit{polar} in \( X \) and is defined by \( A^o := A^* \cap X \). Similarly, if \( A \) is a subset of \( X \) or \( \hat{G} \), its \textit{polar} \( A^q \) in \( G \) is defined by \( A^q := \{ g \in G : |h(g)| \leq \frac{1}{r} \forall h \in A \} \). If \( A \) is a subset of \( G \), it is said to be \textit{quasi-convex} if \( A = A^q \) (or \( A^p \)). Alternatively, if \( A \) is a subset of \( G \) (or \( X \)), we will refer to \( A^p \) (or \( A^q \)) as the \textit{quasi-convex hull} of \( A \). If \( t \) has a neighborhood base at \( 0_G \) consisting of quasi-convex sets, then we say that \( t \) is \textit{locally quasi-convex}.

\textbf{Theorem (2.1).} Let \( A \) be a subset of \( G \).

1. \( A \) is quasi-convex if and only if for every \( g \in G \setminus A \) there is \( h \in A^p \) such that \( |h(g)| > \frac{1}{r} \).
2. The smallest quasi-convex subset containing \( A \) is \( A^q \).

\textbf{Proof.} This is found in Banaszczyk \([2]\) (§1). \(\blacksquare\)

A similar result is obtained for \( A \subset X \).

Each element \( g \in G \) defines a homomorphism \( \phi_g : X \rightarrow T \), defined by \( \phi_g(h) := h(g) \). The function \( \phi_g \) is called the \textit{evaluation function defined by} \( g \). Recall that \( \sigma (\mathcal{C}) \) denotes the weakest topology on \( G \) (or \( X \)) that makes the elements of \( X \) (or \( \mathcal{C} \)) continuous. Notice that \((G, \sigma) = X \) \([8]\), and \((X, \sigma) = G \) \([5] \) (3.8) or Raczkowski and Trigos-Arrieta \([11]\)). Note as well that \((\hat{G}, \sigma) \) is compact (Hewitt and Ross \([9]\) (23.17)), and \((X, \sigma) \) is a dense subgroup of \((\hat{G}, \sigma) \).
The concept of polar is generalized as follows: for any \(0 < \varepsilon \leq \frac{1}{4}\) and \(A \subseteq G\), we define \((A, \varepsilon) := \{h \in X : |h(g)| \leq \varepsilon \forall g \in A\}\). Note that \(A^p = (A, \frac{1}{4})\). If \(A \subseteq X\), we similarly set \((A, \varepsilon) := \{g \in G : |h(g)| \leq \varepsilon \forall h \in A\}\).

**Lemma (2.2).** For any \(0 < \varepsilon \leq \frac{1}{4}\) and \(A \subseteq G\), \((A, \varepsilon)\) is \(\varepsilon\)-closed.

**Proof.** If \(h_0 \not\in (A, \varepsilon)\), then there is \(a \in A\) such that \(|h_0(a)| > \varepsilon\). Take \(\eta := (|h_0(a)| - \varepsilon)/2\) and define \(\phi_a : X \rightarrow T\) by \(\phi_a(h) := h(a)\). Clearly \(\phi_a\) is \(\varepsilon\)-continuous. Consider \(V := \phi_a^{-1}[B_\eta(h_0(a))]\), where \(B_\eta(h_0(a))\) is the ball in \(T\) around \(h_0(a)\) of radius \(\eta\). Then \(V\) is a \(\varepsilon\)-neighborhood of \(h_0\) which misses \((A, \varepsilon)\).

A similar result is obtained for \(A \subseteq X\).

The following is Lemma 1.2 of [2].

**Lemma (2.3).** Let \(\phi : G \rightarrow T\) be a homomorphism, and assume that \(g \in G\) and \(m \in \mathbb{N}\) are such that \(|\phi(kg)| < \frac{1}{4}\) for \(k = 1, \ldots, m\). Then \(\phi(mg) = m\phi(g)\). □

Recall that \(B \subseteq X\) is \(t\)-equicontinuous at \(g_0 \in G\) if for arbitrary \(\varepsilon > 0\), there is a \(t\)-neighborhood \(U\) of \(g_0\) such that if \(g \in U\), then \(|h(g)| < \varepsilon\) for every \(h \in B\). For topological groups, equicontinuity at one point implies equicontinuity at every point. The following result is related to Corollaries 1.2 and 1.3 of [5].

**Theorem (2.4).** If \(A\) is a subset of \(G\), then

1. If \(A\) is quasi-convex, then \(A\) is \(\sigma\)-closed, and thus \(t\)-closed.
2. \(A^p = A^{p_{\Phi}}\), hence \(A^p\) is always quasi-convex.
3. Assume that \(0 < \varepsilon \leq \frac{1}{4}\). If \(U \subseteq t\), then \((U, \varepsilon)\) is \(t\)-equicontinuous, hence \(\varepsilon\)-compact.
4. Let \(B \subseteq X\). If it is \(t\)-equicontinuous and \(0 < \varepsilon \leq \frac{1}{4}\), then \((B, \varepsilon)\) is a \(t\)-neighborhood of \(0_G\); thus \(B\) is \(t\)-equicontinuous if and only if \(B^{p_{\Phi}}\) is \(t\)-equicontinuous. In particular \(t\)-equicontinuous subsets of \(X\) are relatively \(\varepsilon\)-compact.

**Proof.** For (1), if \(A\) is quasi-convex, then \(A = (A^p, \frac{1}{4})\). Now apply Lemma (2.2). For (2), notice that \(A^p \subseteq (A^p)^{p_{\Phi}}\). On the other hand, if \(A \subseteq A^{p_{\Phi}}\), then \((A^{p_{\Phi}})^p \subseteq A^p\). To see (3), let \(\eta > 0\) be given, and choose \(n \in \mathbb{N}\) such that \(\varepsilon/n < \eta\). Let \(V\) be a \(t\)-neighborhood of \(0_G\) such that \(kV \subseteq U\) for \(k = 1, \ldots, n\). If \(h \in (U, \varepsilon)\) and \(v \in V\), then \(|h(v)| < \eta\). Otherwise an application of Lemma (2.3) yields \(nv \in U\), thus \(\varepsilon \geq |h(nv)| = n|h(v)| \geq n\eta > \varepsilon\), a contradiction. Thus \((U, \varepsilon)\) is equicontinuous. To see that it is \(\varepsilon\)-compact, notice that \((X, \varepsilon)\) can be identified with a subspace of \(T^G\). Thus we will show that \((U, \varepsilon)\) is closed in \(T^G\). Let \(f\) be in the closure of \((U, \varepsilon)\) in \(T^G\), and let \(\langle h_\lambda \rangle_{\lambda \in \Lambda}\) be a net of elements in \((U, \varepsilon)\) converging to \(f\). It is straightforward to show that \(f\) is a homomorphism such that \(|f(g)| \leq \varepsilon\) for all \(g \in A\). We proceed to show that it is \(t\)-continuous. Let \(\eta > 0\) be given. By the equicontinuity of \((U, \varepsilon)\), choose a \(t\)-neighborhood \(V\) of \(0_G\) with \(V \subseteq \bigcap_{\lambda \in \Lambda} h_\lambda^{-1}([-\frac{1}{4}, \frac{1}{4}])\) and notice that the latter intersection is contained in \(f^{-1}([-\eta, \eta])\). Finally, to see (4), notice that if \(0_G \in U \subseteq t\) satisfies \(|h(g)| < \varepsilon\) for arbitrary \(g \in U\) and \(h \in B\), then \(U \subseteq (B, \varepsilon)\). Hence \(B \subseteq (B, \varepsilon)^p \subseteq U^p\) and (3) imply the last assertions. □
Remark (2.5). Noble [10] originally proved (3) of Theorem (2.4); its proof as given in Proposition 1.5 of [2] is wrong as the author assumes that $(X, c)$ can be identified with a closed subspace of $T^G$. This barely happens [11]. See also Bruguera [4] §3.1. Statement (4) (together with (3)) improves Corollary 1.3 of [5]. We will strengthen (4) in Corollary (3.3) infra.

Casco et al [5] give a way of producing locally quasi-convex topologies in a topological group $(G, t)$: Let $B \subset X$ be nonempty and $n \in \mathbb{N}$. Denote by $(n)B$ the set \{nh : h \in B\}.

Definition (2.6). A nonempty family $\mathcal{G}$ of subsets of $X$ is called well-directed if the following conditions hold:

(a) For $B_1, B_2 \in \mathcal{G}$, there exists $B_3 \in \mathcal{G}$ such that $B_1 \cup B_2 \subset B_3$.

(b) For $B \in \mathcal{G}$ and $n \in \mathbb{N}$, there exists $A \in \mathcal{G}$ such that $(n)B \subset A$.

For example, the family $\mathcal{F}$ of the finite subsets of $X$ is clearly well-directed. Given a family $\mathcal{G}$ of subsets of $X$, supply $G$ with the topology of uniform convergence on the elements of $\mathcal{G}$ by considering the evaluation maps defined by the elements of $G$. This is a group topology on $G$ which we will denote by $\tau_\mathcal{G}$, and a neighborhood subbase for $0_G$ in this topology is given by the family \{(A, \varepsilon) | A \in \mathcal{G}, \varepsilon > 0\}. Denote by $\mathcal{G}_0$ the minimal well-directed family of subsets of $X$ containing $\mathcal{G}$.

Proposition (2.7). A neighborhood base of $0_G$ for $\tau_\mathcal{G}$ is given by $\{A^d | A \in \mathcal{G}_0\}$. Hence $(G, \tau_\mathcal{G})$ is a locally quasi-convex topological group.

Proof. This is part of Proposition 3.4 in [5].

Corollary (2.8). If $\mathcal{G}$ is well-directed, then $\mathcal{G}_0 := \{A^d | A \in \mathcal{G}\}$ forms a neighborhood base of $0_G$ for $\tau_\mathcal{G}$.

We will return to this issue in Proposition (3.7) infra. It is straightforward to check:

Corollary (2.9). If $(G, \lambda)$ is a topological group with character group $Y$, denote by $\mathcal{L}$ the family of $\lambda$-equicontinuous subsets of $Y$. Then:

1. $\mathcal{L}$ is well-directed,
2. $\mathcal{L}_0$ forms a neighborhood base of $0_G$ for $\tau_\mathcal{L}$,
3. $\sigma \subseteq \tau_\mathcal{L} \subseteq \lambda$, and
4. $\lambda$ is quasi-convex if and only if $\lambda = \tau_\mathcal{L}$.

Results (3) & (4) of Theorem (2.4) are (a) & (b) respectively of Proposition 3.9 of [5].

3. Main results

In this section we characterize the existence of the Mackey topology of a locally quasi-convex topological group.

Let $\mathcal{M} := \{A \subset X | (A, \varepsilon)^\lor = (A, \varepsilon)^\lor \text{ whenever } 0 < \varepsilon \leq \frac{1}{4}\}$.

Since $(\hat{G}, c)$ is compact, Lemma (2.2) implies that the members of $\mathcal{M}$ are relatively $c$-compact. We point out right now that although $\mathcal{M}$ is closed under taking quasi-convex hulls (Proposition (3.6)), sometimes there are $c$-compact
quasi-convex sets that do not belong to $\mathfrak{M}$ (next section). We will show that $G$ attains its Mackey topology $\mu$ with character group $X$ if and only if $\mathfrak{M}$ is well-directed; and if so, $\mu = \tau_{\mathfrak{M}}$. To do so we need to establish some properties of $\mathfrak{M}$. Notice that if $\lambda$ is any other group topology on $G$ compatible with $t$, then $(G, t)^{\lambda} = X = (G, \lambda)^{\lambda}$. Hence it is important to realize that $\mathfrak{M}$ does not depend on whether we choose $\lambda$ or $t$ in its definition, as $\mathfrak{M}$ depends only on $X$. Also note that $(A, \varepsilon)^{\mu} \subseteq (A, \varepsilon)^{\Phi}$ is always valid, hence if $A \in \mathfrak{M}$ and $B \subseteq A$, then $B \in \mathfrak{M}$. It is also worth to notice that $\mathfrak{F} \subseteq \mathfrak{M}$, where $\mathfrak{F}$ stands for the family of finite subsets of $X$.

**Proposition (3.1).** Let $(G, t)$ be a topological group, not necessarily locally quasi-convex. If $U$ is a neighborhood of $0_G$ and if $0 < \eta \leq \frac{1}{b}$ is given, then $(U, \eta) \subseteq \mathfrak{M}$.

**Proof.** It is enough to show that $U^p \in \mathfrak{M}$, as $(U, \eta) \subseteq U^p$. Consider $0 < \varepsilon \leq \frac{1}{b}$. If $h \in (U^p, \varepsilon)^{\Phi}$, we need to show that $h$ is $t$-continuous.

Let $0 < \delta \leq \frac{1}{b}$ and choose $n \in \mathbb{N}$ such that $\frac{\varepsilon}{n} < \delta$. By (3) of Theorem (2.4) $U^p$ is $t$-equicontinuous, hence there exists a $t$-neighborhood $V$ of $0_G$ such that $V \subseteq (U^p, \varepsilon)$. By continuity of the group operation, there exists a $t$-neighborhood $W$ of $0_G$ such that $W, 2W, \ldots, nW \subseteq V$. For $g \in W \subseteq V \subseteq (U^p, \varepsilon)$ we have that $|h(g)| \leq \varepsilon$, $|h(2g)| \leq \varepsilon$, $\cdots$, $|h(ng)| \leq \varepsilon$, hence by Lemma (2.3), it follows that $|h(g)| \leq \frac{\varepsilon}{n} < \delta$, i.e., $W \subseteq h^{-1}[-\delta, \delta]$ which proves that $h$ is $t$-continuous. $\square$

This sharpens (3) of Theorem (2.4) supra.

**Corollary (3.2).** If $(G, t)$ is a locally quasi-convex topological group, then $\sigma \leq t \leq \tau_{\mathfrak{M}}$. If $\lambda$ is any other group topology on $G$ compatible with $t$, then $\sigma \leq \lambda \leq \tau_{\mathfrak{M}}$.

**Proof.** Take a neighborhood base $B$ of $0_G$ consisting of quasi-convex subsets. If $U \in B$, then $U = U^{p\eta} = (U^p, \frac{1}{b})$, hence $U \in \tau_{\mathfrak{M}}$. $\square$

**Corollary (3.3).** If $(G, t)$ is a locally quasi-convex topological group and $B \subseteq X$ is $t$-equiuniform, then $B \in \mathfrak{M}$.

**Proof.** Follows from (4) of Theorem (2.4). $\square$

**Corollary (3.4).** Let $\lambda$ be a group topology on $G$ and assume that $\Sigma$ stands for the $\lambda$-equiuniform subsets of $(G, \lambda)^{\lambda}$. If $\lambda$ is compatible with $t$, then $\Sigma \subseteq \mathfrak{M}$.

The last two corollaries sharpen (c) of Proposition 3.9 of [5].

We now proceed to study if the sufficiency of the last condition holds.

**Proposition (3.5).** Assume that $A \subseteq X$. Let $0 < \varepsilon \leq \frac{1}{b}$ and $n \in \mathbb{N}$. If $g \in (A, \varepsilon)$, then $g, 2g, \cdots, ng \in (A, \varepsilon)$.

**Proof.** Let $h \in A$ and $g \in (A, \frac{\varepsilon}{n}) \subseteq (A, \varepsilon)$. Then $|h(g)| \leq \frac{\varepsilon}{n} < \frac{\varepsilon}{n-1} < \cdots < \frac{\varepsilon}{n} < \varepsilon$. This implies that $|h(n)g| = n|h(g)| \leq \varepsilon$, $|h((n-1)g)| = (n-1)|h(g)| \leq \varepsilon$, $\cdots$, $|h(2g)| = 2|h(g)| \leq \varepsilon$, $|h(g)| \leq \varepsilon$. $\square$

Recall that if $\mathfrak{G}$ denotes a non-empty family of non-empty subsets of $X$, then $\mathfrak{G}^d := \{A^d \mid A \in \mathfrak{G}\}$; we also set $\mathfrak{G}^\Phi := \{A^\Phi \mid A \in \mathfrak{G}\}$. 

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PROPOSITION (3.6). \( \mathcal{M}^\varphi \subset \mathcal{M} \).

Proof. Let \( A \in \mathcal{M} \). We first show that for \( 0 < \varepsilon \leq \frac{1}{4} \) there is \( n \in \mathbb{N} \) such that \( (A, \frac{\varepsilon}{n}) \subset (A^\varphi, \varepsilon) \). For this, let \( n \in \mathbb{N} \) such that \( \frac{1}{4n} < \varepsilon \). If \( g \in (A, \frac{\varepsilon}{n}) \), by Proposition (3.5) it follows that \( g, 2g, \ldots, ng \in (A, \varepsilon) \subset (A, \frac{1}{2}) = A^\varphi \). Hence, if \( h \in A^\varphi = (A^\varphi, \frac{1}{2}) \), then \( |h(g)| \leq \frac{1}{2}, |h(2g)| \leq \frac{1}{2}, \ldots, |h(n g)| \leq \frac{1}{2} \). By Lemma (2.3) it follows that \( |h(g)| \leq \frac{1}{4n} < \varepsilon \), that is \( (A, \frac{\varepsilon}{n}) \subset (A^\varphi, \varepsilon) \).

Now, it follows that \( (A^\varphi, \varepsilon)^\varphi \subset (A, \frac{\varepsilon}{n})^\varphi \), but \( (A, \frac{\varepsilon}{n})^\varphi \subset \mathcal{X} \) because \( A \in \mathcal{M} \). This proves that \( (A^\varphi, \varepsilon)^\varphi = (A^\varphi, \varepsilon)^\varphi \) for \( 0 < \varepsilon \leq \frac{1}{4} \), i.e., \( A^\varphi \in \mathcal{M} \) for \( A \in \mathcal{M} \).

If \( B \subset X \) and \( n \in \mathbb{N} \), we let \( (1/n)B := \{ x \in X : nx \in B \} \). If \( C \subset G \), it follows that \( (nC)^\varphi = (1/n)(C^\varphi) \) (cf. [4] (4.1.2(iii))).

PROPOSITION (3.7). Let \( \mathcal{G} \) be a non-empty family of subsets of \( X \) satisfying \( \mathcal{G}^\varphi \subset \mathcal{G} \). Then \( \mathcal{G} \) is well-directed if and only if \( \mathcal{G}^\varphi \) is a fundamental system of neighborhoods for \( 0_G \) in \( \tau_{\mathcal{G}} \).

Proof. Necessity of the second condition is Corollary (2.8). To see its sufficiency notice that, as a group topology, \( \tau_{\mathcal{G}} \) must satisfy the topological group axioms. If \( \{ A_1, A_2 \} \subset \mathcal{G} \), then there must be \( A_3 \in \mathcal{G} \) such that \( A_3^\varphi \subset A_1^\varphi \cap A_2^\varphi \); the latter equals \( (A_1 \cup A_2)^\varphi \) ([4] (4.1.2(ii))), hence \( A_1 \cup A_2 \subset (A_1 \cup A_2)^\varphi \subset A_3^\varphi \), which, by hypothesis, belongs to \( \mathcal{G} \), hence the first requirement for \( \mathcal{G} \) to be well-directed is satisfied. To see the second requirement, let \( A \in \mathcal{G} \) and \( n \in \mathbb{N} \). Choose \( B \in \mathcal{G} \) such that \( n(B^\varphi) \subset A^\varphi \), hence \( A \subset A^\varphi \subset (n(B^\varphi))^\varphi = (1/n)(B^\varphi) = \{ x \in X : nx \in B^\varphi \} \), and thus \( (n)A \subset B^\varphi \), which by hypothesis, belongs to \( \mathcal{G} \).

For example, Aussenhofer [1] (7.11) has proved that \( \mathfrak{G}^\varphi \subset \mathfrak{G} \), where \( \mathfrak{G} \) denotes the finite subsets of \( X \), hence \( \mathfrak{G}^\varphi \) is a fundamental system of neighborhoods for \( 0_G \) in \( \tau_{\mathcal{G}} = \sigma \).

PROPOSITION (3.8). The family \( \mathcal{M} \) is well-directed if and only if \( \mathcal{M}^\varphi \) is a fundamental system of neighborhoods for \( 0_G \) in \( \tau_{\mathcal{M}} \).

Proof. Follows from Propositions (3.6) and (3.7).

PROPOSITION (3.9). If \( (G, t) \) is a locally quasi-convex topological group and \( \mathcal{M} \) is well-directed, then \( (G, t)^\varphi = (G, \tau_{\mathcal{M}})^\varphi \), i.e., \( \tau_{\mathcal{M}} \) is compatible with \( t \).

Proof. Since \( t \leq \tau_{\mathcal{M}} \) from Corollary (3.2), clearly we have \( (G, t)^\varphi \subset (G, \tau_{\mathcal{M}})^\varphi \). If \( h \in (G, \tau_{\mathcal{M}})^\varphi \), then by Proposition (3.8), there is \( A \in \mathcal{M} \) and \( 0 < \varepsilon \leq \frac{1}{4} \) with \( (A, \varepsilon) \subset h^{-1}([-\frac{1}{4}, \frac{1}{4}]) \). Hence \( h \in (A, \varepsilon)^\varphi = (A, \varepsilon)^\varphi \subset X \), since \( A \in \mathcal{M} \).

Remark (3.10). Propositions (3.1) and (3.9) above are valid if we substitute \( (G, t) \) by \( (G, \lambda) \) where \( \lambda \) is any other group topology on \( G \) compatible with \( t \), that is, if \( (G, \lambda)^\varphi = X \).

We summarize the contents of this section as follows:
THEOREM (3.11). (Mackey-Arens Theorem for locally quasi-convex topological groups.) Let \((G, t)\) be a MAP topological group (not necessarily locally quasi-convex) with character group \(X\). Consider

\[
\mathcal{M} := \left\{ A \subset X \mid (A, \varepsilon)^\supseteq (A, \varepsilon)^\subset \text{ whenever } 0 < \varepsilon \leq \frac{1}{4} \right\}.
\]

If \(\lambda\) is a locally quasi-convex group topology on \(G\), assume that \(\Sigma\) stands for the \(\lambda\)-equicontinuous subsets of \((G, \lambda)^\cap\), and let \(\sigma\) denote the weakest topology on \(G\) that makes the elements of \(X\) continuous.

1. If \(\lambda\) is compatible with \(t\), then \(\sigma \leq \lambda \leq \tau_\mathcal{M}\), and \(\Sigma \subseteq \mathcal{M}\). 2. The following assertions are equivalent:
   (a) The family \(\mathcal{M}\) is well-directed,
   (b) \(\mathcal{M}^s\) is a fundamental system of neighborhoods for \(O_G\) in \(\tau_\mathcal{M}\), and
   (c) if \(\sigma \leq \lambda \leq \tau_\mathcal{M}\), then \(\lambda\) is compatible with \(t\).

Proof. Statement (1) are Corollaries (3.2) and (3.4). To see (2), notice that (a) \(\iff\) (b) is Proposition (3.8), whereas (b) \(\implies\) (c) follows from Proposition (3.9), and the fact that the character group of \((G, \sigma)\) is precisely \(X\). Finally, if (c) holds, then \((G, \tau_\mathcal{M})^\cap = \mathcal{M}\), hence \(\mathcal{M}\) is equal to the family of \(\tau_\mathcal{M}\)-equicontinuous subsets of \((G, \tau_\mathcal{M})^\cap\), which by (1) of Corollary (2.9), forms a well-directed family. \(\Box\)

The above theorem gives a positive answer to Question 2 in [5] (pag. 275), i.e., when the Mackey topology exists, it has the form \(\tau_\mathcal{M}\) for an appropriate family \(\mathcal{G}\) of subsets of \(X\) (take \(\mathcal{G} = \mathcal{M}\)). It also contributes partially answer to question 1 in the same article. Nevertheless Question 1 remains open in its entirety:

Question (3.12). Given a MAP topological group \((G, t)\), is there a finest locally quasi-convex topology \(\mu\) on \(G\) compatible with \(t\)?

Remark (3.13). In [5] (4.1) it is proved that if a topological group \((G, t)\) is \(g\)-barrelled, i.e., it is such that every compact subset of \((X, \xi)\) is \(t\)-equicontinuous, then its Mackey topology \(\mu\) exists and \(\mu \leq t\). Such groups contain the classes of locally compact and completely metrizable groups. Since we know that \(\mu = \tau_\mathcal{M}\), it follows that for these groups, \(\tau_\mathcal{M} \leq t\), and the equality follows if \(t\) is locally quasi-convex, as it is the case when \(t\) is locally compact.

It would be interesting to obtain a useful characterization of the family \(\mathcal{M}\).

4. Examples

In [5] it is asked if given a topological group \((G, t)\): (a) whether the family \(\mathcal{G}_{qc}\) of all compact quasi-convex subsets of \((X, \xi)\) is always well-directed (p. 271), (b) whether \(\mu\) is always the topology \(\tau_{qc}\) of the uniform convergence on the elements of \(\mathcal{G}_{qc}\) (Remark 9), and (c) whether \(\mu\) and \(\tau_{qc}\) are always compatible (beginning of §4). The answer to these questions is “not always” as Examples (4.1), (4.2), (4.3), (4.5) and (4.6) in this section show. We also give an application of Theorem (3.11) (Example (4.4)) proving that for totally bounded countable metric groups \((G, t)\) of bounded order, the family \(\mathcal{M}\) coincides with the finite subsets of the character group, hence \(t\) in this case coincides with the Mackey topology of \((G, t)\).
Let $\Gamma$ be any finite (discrete) group, and consider $X := \oplus_{n<\omega} \Gamma$ equipped with the topology inherited from the product $\Gamma^\omega$. For $x = (x_n)_{n<\omega} \in X$, the \textit{support} of $x$, denoted by $\text{supp}(x)$, is defined by

$$\text{supp}(x) := \{ n < \omega : x_n \neq 0 \}.$$  

Then $K := \{ x \in X : \text{supp}(x) \leq 1 \}$ is compact. For, let $U$ be a neighborhood of $0_X$. Then there are $\{n_1, \ldots, n_k\} \subset \omega$ such that the basic open set $V := \{(x_n)_{n \in \omega} \in X : x_{n_j} = 0 \forall j \in \{1, \ldots, k\}\}$ is contained in $U$ with $0_X \in V$. But then clearly $K \setminus V$ is finite. Note that $K \setminus \{0_X\}$ is discrete: if $x = (x_n)_{n \in \omega} \in K \setminus \{0_X\}$, let $V_x := \{(y_n)_{n \in \omega} : x_m = y_m, m \in \text{supp}(x)\}$. Then $V_x$ is a neighborhood of $x$ and $V_x \cap K = \{x\}$. This construction appears in Comfort, Raczkowski, and Trigos-Arrieta [7] (3.9); see also [6].

\textbf{Example} (4.1). Consider $\mathbb{Z}_4$ identified with $\{0, \pm \frac{1}{2}, \pm \frac{3}{2}\}$, and with operation modulo 1. Then $\Gamma := \mathbb{Z}_4^\omega = \{0, \pm1, \pm2\}$ acts on $\mathbb{Z}_4$ by multiplication modulo 1. Consider $G := \oplus_{n<\omega} \mathbb{Z}_4$ equipped with the topology inherited from the product $(\mathbb{Z}_4)^\omega$. Then its character group is $X = \oplus_{n<\omega} \Gamma = \oplus_{n<\omega} \mathbb{Z}_4^\omega$, and $\varsigma$ on $X$ is the topology inherited from the product of $\mathbb{Z}_4^\omega$ [11]. It follows that $F := (K \setminus \{x : 2 \not\in \text{supp}(x)\}) \cup \{0_X\}$ is $\varsigma$-compact and quasi-convex. Indeed, straightforward calculations show that $F^d = \oplus_{n<\omega} \{0, \pm \frac{1}{2}\}$, hence $F^\Phi = F$. However $(2)F = \{2x : x \in F\} = (K \setminus \{x : \pm1 \not\in \text{supp}(x)\}) \cup \{0_X\}$, thus $((2)F)^d = \oplus_{n<\omega} \{0, \frac{3}{2}\}$, and hence $((2)F)^\Phi = \oplus_{n<\omega} \{0, \frac{3}{2}\}$ which clearly is not contained in any $\varsigma$-compact subset of $X$. Thus requirement (b) of being well directed is not held by this particular $\mathcal{Q}_{\varsigma_{qc}}$.

\textbf{Example} (4.2). Consider next $\mathbb{Z}_5$ identified with $\{0, \pm \frac{1}{2}, \pm \frac{3}{2}\}$, and with operation modulo 1. If $G := \oplus_{n<\omega} \mathbb{Z}_5$ is equipped with the topology inherited from the product $(\mathbb{Z}_5)^\omega$, then its character group is $X = \oplus_{n<\omega} \Gamma = \oplus_{n<\omega} \mathbb{Z}_5^\omega$, with $\varsigma$ on $X$ being the topology inherited from the product of $(\mathbb{Z}_5)^\omega$ [11]. The sets $F_1 := (K \setminus \{x : \pm2 \not\in \text{supp}(x)\}) \cup \{0_X\}$, and $F_2 := (K \setminus \{x : \pm1 \not\in \text{supp}(x)\}) \cup \{0_X\}$ are $\varsigma$-compact and quasi-convex. Indeed, straightforward calculations show that $F_1^d = \oplus_{n<\omega} \{0, \pm \frac{1}{2}\}$, and $F_2^d = \oplus_{n<\omega} \{0, \pm \frac{3}{2}\}$; hence $F_1^\Phi = F_1$ and $F_2^\Phi = F_2$. However $F_1 \cup F_2 = K$, hence $(F_1 \cup F_2)^d = \{0_X\}$, and thus $(F_1 \cup F_2)^\Phi = X$ which clearly cannot be contained in any $\varsigma$-compact subset of $X$. Thus requirement (a) of being well directed is not held by this particular $\mathcal{Q}_{\varsigma_{qc}}$.

\textbf{Example} (4.3). Let us point out that by taking $G := \oplus_{n<\omega} (\mathbb{Z}_4 \times \mathbb{Z}_5)$ equipped with the topology inherited from the product $(\mathbb{Z}_4 \times \mathbb{Z}_5)^\omega$, then the family $\mathcal{Q}_{\varsigma_{qc}}$ of all compact quasi-convex subsets of $(X_\varsigma, \varsigma)$ will not hold either one of the properties for being well directed. We leave the details to the reader. Examples (4.1), (4.2) and (4.3) take care of (a).

We now give an application of Theorem (3.11), showing that some noncomplete metric groups already carry their Mackey topology. This is related with the results of the fourth section in [5].

\textbf{Example} (4.4). Let $(G, t)$ be a totally bounded group of bounded order such that $\vert X \vert < c$. Notice that $t = \sigma$ [8]. The family $\mathcal{F}$ of the finite subsets of $X$ is clearly well-directed. We will prove that $\mathcal{M} = \mathcal{F}$. We only need to show the contention $\mathcal{M} \subseteq \mathcal{F}$. Let $F \subseteq X$ be an infinite set. Choose an integer $n \geq 4$
such that \( ng = 0_G \) for every \( g \in G \). Notice that the elements in \( X \) also satisfy \( nx = 0_X \) for every \( x \in X \). Take a positive \( \varepsilon < 1/n \). Then if \( x \in F \) and \( g \in (F, \varepsilon) \), then \( x(g) = 0 \). Hence if \( F \in \mathcal{M} \), then \( (F, \varepsilon)^\circ = (F, \varepsilon)^\circ \subseteq X \) would contain the closure in \( \widehat{G} \) of the group generated by \( F \). Since infinite compact groups have cardinality at least \( \varepsilon \), we conclude that \( \mathcal{M} \subseteq \mathcal{G} \), as required. By Theorem (3.11), \( t = \sigma \) is the Mackey topology on \((G, t)\).

Hence any metric, totally bounded group of bounded order carries already its Mackey topology. In particular the topologies of the groups presented in Examples (4.1), (4.2) and (4.3) equal \( \mu \).

**Example** (4.5). Notice that in Example (4.2) above \( \{F_1, F_2\} \subset \mathfrak{S}_{qc} \), hence \( \{0_G\} = F_1^\circ \cap F_2^\circ \subseteq \tau_{qc} \Rightarrow \tau_{qc} = d \). Thus \( X \neq \widehat{G} \), it follows that \( \tau_{qc} \) is not compatible with \( \mu \). This takes care of (c). Moreover, since \((G, \mu) = X \) whereas \((G, \tau_{qc}) = \widehat{G} \), it follows that \( \mu \neq \tau_{qc} \), taking care of (b).

**Example** (4.6). We finally notice that taking \( 0 < \varepsilon < \frac{1}{n} \), each of the sets \( F \) defined above are such that \( (F, \varepsilon)^\circ \neq (F, \varepsilon)^\circ \). For, \( (F, \varepsilon) = \{0_G\} \), hence \( (F, \varepsilon)^\circ = X \) whereas \( (F, \varepsilon)^\circ = \widehat{G} \). Since \( \mu \) is the topology of the uniform convergence on \( \mathcal{M} \), this is another way to answer (b).

In [5] it is said that a topological group \((G, t)\) is **pre-Mackey** if the elements of \( \mathfrak{S}_{qc} \) are \( t \)-equicontinuous. If so, then \( \mu = \tau_{qc} \). The examples in this section are clearly not pre-Mackey. Notice as well that these examples show that the requirement of being Baire cannot be dropped in [5] (1.6).

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