CALIFORNIA STATE UNIVERSITY, BAKERSFIELD

Lee Webb Math Field Day 2014

Team Medley, Junior-Senior Level

Each correct answer is worth ten points. Answers require justification. Partial credit may be given. Unanswered questions are given zero points.

You have 50 minutes to complete the Exam. When the exam is over, give only one set of answers per team to the proctor. Multiple solutions to the same problem will invalidate each other.

Elegance of solutions may affect score and may be used to break ties.

All calculators, cell phones, music players, and other electronic devices should be put away in backpacks, purses, pockets, etc. Leaving early or otherwise disrupting other contestants may be cause for disqualification.
1. (a) In Springfield, all streets run North-South or East-West. How many ways are there to get from Jimmy's house to Timmy's house? Both houses are at intersections and Timmy's house is 5 blocks to the north and 5 blocks to the east of Jimmy's.

(b) How many of these ways stay north of the direct, “as the crow flies,” route.

Solution:

(a) A path from Jimmy's to Timmy's, is a string of 5 N's and 5 E's. To count these we only have to choose which 5 of the 10 places have N's. The answer therefore is \( \binom{10}{5} = \frac{10!}{5! \cdot 5!} = 252 \).

(b) Now we have to count how many of the strings of N's and E's are such that, as we read them from left to right, have the property that the number of N's is always greater than the number of E's (until the last letter, which must be E, is counted). We can organize the list of such strings, by how many E's are among the 5 N's.

NNNNNNEEEEEE - no E's before the 5th N 1 string
NN*N*N*N*NEEEE - an E could sub for any of the *'s 3 strings
2 E's among first 5 N's
NNEN*N*N  E could sub for any * 2 strings
NNN*N*N  2 E's in either place, or 1 each 3 strings
NNNNEENEEE 1 string
3 E's among first 5 N's
NNEN*N*N  1 E for each *, or 2 for second * 2 strings
NNN*N*N  1 or 2 E's for first *, or 3 for second 3 strings

Total = 15 possibilities.

2. How many strings of 10 vowels can be made, if every vowel (a,e,i,o,u) has to be used at least once?

Solution:

Let U be the set of all possible strings, A = set of strings missing “a”, B = set of strings missing “b”, etc. We need to determine the number of
elements in the complement of \( A \cup B \cup C \cup D \cup E \). By the inclusion-exclusion principle, we have:

\[
|A \cup B \cup C \cup D \cup E| = |U| - (|A|+|B|+|C|+|D|+|E|) + (|AB|+|AC|+...+|DE|) \ (all \ pairs) \\
- (|ABC|+...+|CDE|) \ (all \ triplets) \\
+ (|ABCD|+|ABCE|+|ABDE|+|ACDE|+|BCDE|) \\
= 5^{10} - 5 \cdot 4^{10} + 10 \cdot 3^{10} - 10 \cdot 2^{10} + 5 \cdot 1^{10} \\
= 5103000
\]

3. Ella and Daniel have made up a game with small stones (they are so deprived – these are the only toys they have). The stones are placed in a pile and they take turns removing stones. At each turn, a player either removes one stone from the pile or removes half the stones (rounding up – if necessary) from the pile. The player who takes the last stone wins. Ella will go first (she always goes first since she's younger). Daniel says, “I get to decide how many stones go in the pile, since you can always win with some numbers of initial stones.” Of which numbers is Daniel speaking?

**Solution:**

Daniel is speaking of the numbers which are winning positions for Ella, call this set \( W \). Clearly, \( 1 \in W \). Ella will definitely lose if the initial number of stones is 2. Call the set of numbers for which Ella will lose \( L \), so we have \( 2 \in L \). Now, \( 3 \in W \) since Ella could take one stone and leave Daniel with 2 stones, which is a losing situation for him. Similarly \( 4 \in W \) since Ella could take half of these and again leave Daniel with 2. Then we see every number has to be in one of these two sets, according the rules:

\[
\begin{align*}
n & \in W \text{ if } n-1 \in L \text{ or } n-[n/2] \in L \text{ and } n \in L \text{ otherwise. Thus, we have} \\
W & = \{1,3,4,6,8,10,11,13,14,15,17,18,19,21,23,...\} \quad \text{and} \\
L & = \{2,5,7,9,12,16,20,22,..\}
\end{align*}
\]

4. Prove the formula: \( \pi = 16 \tan^{-1}\left(\frac{1}{5}\right) - 4 \tan^{-1}\left(\frac{1}{239}\right) \).

**Solution:**
We will need the identity
\[ \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \]
from which we also get
\[ \tan 2A = \frac{2\tan A}{1 - \tan^2 A}. \]

Let \( \theta = \tan^{-1}(1/5) \). The given formula is equivalent to
\[ \frac{\pi}{4} = 4\theta - \tan^{-1}\left(\frac{1}{239}\right) \iff 1 = \tan\left(4\theta - \tan^{-1}\left(\frac{1}{239}\right)\right) = \frac{\tan 4\theta - 1/239}{1 + \tan 4\theta / 239} = \frac{239 \tan 4\theta - 1}{239 + \tan 4\theta}. \]

But then we also have, \( \tan 2\theta = \frac{2/5}{24/25} = \frac{5}{12} \) and
\[ \tan 4\theta = \tan(2(2\theta)) = \frac{2\tan 2\theta}{1 - \tan^2 2\theta} = \frac{5/6}{119/144} = \frac{120}{119}. \]
Thus,
\[ \frac{239 \tan 4\theta - 1}{239 + \tan 4\theta} = \frac{239 \cdot 120 / 119 - 1}{239 + 120 / 119} = \frac{239 \cdot 120 - 119}{239 \cdot 119 + 120} = \frac{239(119 + 1) - 119}{239 \cdot 119 + 120} = 1. \]

5. Describe the set of all points that are equidistant from two circles. The circles have radii 1 and 3 and are centered at (0,0) and (6,0), respectively. Repeat in the case that the second circle is centered at (2,0).

Solution:

First case: Let \( P \) be such a point and let \( d_1 \) be the distance from \( P \) to the origin, and \( d_2 \) be the distance from \( P \) to the point (6,0). Then, we have
\[ d_1 - 1 = d_2 - 3 \iff d_2 - d_1 = 2. \]
This is the locus definition of (one branch of) a hyperbola with foci at (0,0) and (6,0).

Second case: Now the first circle is inside the second and they are mutually tangent at (-1,0). Let \( d_3 \) be the distance from \( P \) to (2,0). This time we have
\[ 3 - d_3 = d_1 - 1 \iff d_1 + d_3 = 4, \]
which is the locus definition of an ellipse.

6. Parallel lines are spaced 2 units apart on a large board. A side-length 1 square is placed randomly on the board. What is the probability that a part of the square touches one of the parallel lines?
Solution:

After placing the square on the board, let $C$ be the center of the square, $D$ be the nearest point to $C$ that is on one of the lines, and $A$ be the corner of the square that is nearest to the line that $D$ is on (ties have probability 0, so we can ignore this possibility – or else just say in case of a tie, pick one of the choices arbitrarily). Let $d$ be the distance from $C$ to $D$ and $\theta$ be angle $ACD$. The square touches the line if $d < \frac{\sqrt{2}}{2} \cos \theta$ (the quantity on the right represents the length of the projection of $CA$ onto line $CD$). Since we have $0 < d < 1$ and $-\pi/4 < \theta < \pi/4$, the probability we seek is:

$$\frac{\int_{-\pi/4}^{\pi/4} \frac{\sqrt{2}}{2} \cos \theta \ d \theta}{\int_{-\pi/4}^{\pi/4} 1 \ d \theta} = \frac{\sqrt{2} \sin \theta}{\pi} \bigg|_{0}^{\pi/4} = \frac{2}{\pi}$$