(1) Use vectors to prove that the line joining the midpoints of 2 sides of a triangle is parallel to the third side and measures half its length.

\[ \vec{EF} = \vec{EA} + \vec{AF} = \frac{1}{2} \vec{BA} + \frac{1}{2} \vec{AC} = \frac{1}{2} (\vec{BA} + \vec{AC}) = \frac{1}{2} \vec{BC} \]

Since \( \vec{EF} = \frac{1}{2} \vec{BC} \), therefore, \( EF \) is parallel to \( BC \) and measures half its length.

(2) Let \( OABCDEFG \) be a cube, labelled as shown in the diagram below. Show that \( O\vec{B} + O\vec{D} + O\vec{F} \) is parallel to \( O\vec{G} \).

\[ O\vec{B} + O\vec{D} + O\vec{F} = O\vec{B} + (O\vec{C} + C\vec{D}) + O\vec{F} = (O\vec{B} + C\vec{D}) + (O\vec{C} + O\vec{F}) = (O\vec{B} + B\vec{G}) + (O\vec{C} + F\vec{G}) = O\vec{G} + O\vec{G} = 2O\vec{G}. \]

Hence \( O\vec{B} + O\vec{D} + O\vec{F} \) is parallel to \( O\vec{G} \).

(3) A constant force with vector representation \( \vec{F} = \langle 5, 4, -3 \rangle \) moves an object along a straight line from the point \( (2, 3, 0) \) to the point \( (4, 6, 8) \). Find the work done, if the distance is measured in meters and the magnitude of the force is measured in newtons.

Distance vector \( \vec{D} = \langle 4 - 2, 6 - 3, 8 - 0 \rangle = \langle 2, 3, 8 \rangle \). Hence

\[ WD = \vec{F} \cdot \vec{D} = \langle 5, 4, -3 \rangle \cdot \langle 2, 3, 8 \rangle = 5(2) + 4(3) - 3(8) = -2 \]

(4) Velocities have both magnitude and direction and thus are vectors. The magnitude of a velocity vector is called its speed. Suppose that a wind is blowing from the direction N45°W at a speed of 40 km/h (this means the direction from which the wind blows is 45° west of the northerly direction). A pilot is steering a plane in the direction of S60°E at an airspeed (speed in the air) of 350 km/h. The true course, or track of the plane is the direction of the resultant velocity vectors of the plane and wind.
The ground speed of the plane is the magnitude of the resultant. Find the true course and ground speed of the plane.

Let \( \mathbf{p} \) be the velocity vector of the plane and \( \mathbf{w} \) be the velocity vector of the plane. Then

\[
\mathbf{p} = 175\sqrt{3} \mathbf{i} - 175 \mathbf{j}
\]

\[
\mathbf{w} = 20\sqrt{2} \mathbf{i} - 20\sqrt{2} \mathbf{j}
\]

The resultant is the vector \( \mathbf{v} = \mathbf{p} + \mathbf{w} = (175\sqrt{3} + 20\sqrt{2}) \mathbf{i} - (175 + 20\sqrt{2}) \mathbf{j} \).

The ground speed of the plane is \( |\mathbf{v}| = |\mathbf{p} + \mathbf{w}| = 388.76 \text{ km/h} \). The direction is S\( \theta \)E where \( \theta \approx 1.021 \approx 58.47^\circ \).

(5) Ropes 3m and 5m in length are fastened to a holiday decoration that is suspended over a town square. The decoration has a mass of 5kg. The ropes, fastened at different heights, make angles of 52° and 40° with the horizontal. Find the tension in each wire and the magnitude of each tension.

![Diagram of ropes and decoration](image)

Let \( \mathbf{T}_3 \) and \( \mathbf{T}_5 \) be the tensions on the 3m and 5m string. Then,

\[
\mathbf{T}_3 = -|\mathbf{T}_3| \cos 52^\circ \mathbf{i} + |\mathbf{T}_3| \sin 52^\circ \mathbf{j}
\]

\[
\mathbf{T}_5 = -|\mathbf{T}_5| \cos 40^\circ \mathbf{i} + |\mathbf{T}_5| \sin 40^\circ \mathbf{j}
\]

The total tension is equal to the force exerted by the decoration, which has a force \( 5g\mathbf{j} = 5(9.8)\mathbf{j} = 49\mathbf{j} \) (in Newtons).

Therefore, \( \mathbf{T}_3 + \mathbf{T}_5 = 49\mathbf{j} \). Resolving in the \( \mathbf{i} \) and \( \mathbf{j} \) direction, we have

\[
-|\mathbf{T}_3| \cos 52^\circ - |\mathbf{T}_5| \cos 40^\circ = 0
\]

\[
|\mathbf{T}_3| \sin 52^\circ + |\mathbf{T}_5| \sin 40^\circ = 49
\]

Solving for \( |\mathbf{T}_3| \) and \( |\mathbf{T}_5| \), we have \( |\mathbf{T}_3| \approx 38N \) and \( |\mathbf{T}_5| \approx 30N \), and

\[
\mathbf{T}_3 \approx -23\mathbf{i} + 30\mathbf{j}
\]

\[
\mathbf{T}_5 \approx 23\mathbf{i} + 19\mathbf{j}
\]

(6) Show that the vector

\[
\mathbf{v} = \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}
\]

is orthogonal to \( \mathbf{a} \). Given vectors \( \mathbf{a} \), \( \mathbf{b} \), can you draw on a diagram what the vector \( \mathbf{v} \) is?

Note that \( \text{proj}_\mathbf{a} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} \), hence \( \mathbf{v} \) is the vector as shown in the diagram (hence it should be orthogonal to \( \mathbf{a} \).
To show the result algebraically, we show that $\mathbf{v} \cdot \mathbf{a} = 0$ as follows

$$\mathbf{v} \cdot \mathbf{a} = \left( \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} \right) \cdot \mathbf{a}$$
$$= \mathbf{b} \cdot \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} |\mathbf{a}|^2$$
$$= \mathbf{a} \cdot \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} |\mathbf{a}|^2$$
$$= \mathbf{a} \cdot \mathbf{b} - |\mathbf{a}|^2 = 0.$$

(7) The *Cauchy-Bunyakovskii-Schwarz* (CBS) inequality states that

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| \ |\mathbf{b}|.$$

(a) Prove the CBS inequality from the definition of the dot product.

Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \ |\mathbf{b}| \cos \theta$. Hence

$$|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}| \ |\mathbf{b}| \ |\cos \theta| \leq |\mathbf{a}| \ |\mathbf{b}|$$

(b) What conditions must satisfy if the CBS inequality becomes a equality?

The inequality becomes a equality when $|\cos \theta| = 1$, that is, $\theta = 0$ or $\pi$. That is, when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

(8) The *Triangle Inequality* states that

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|.$$

(a) Give a geometric interpretation why the above inequality is called the triangle inequality.

(b) Prove $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$

Since $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$, we must have

$$|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$$

(c) Use the dot product, the CBS inequality, and the above to show the triangle inequality.
\[ |\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) \]
\[ = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \]
\[ = |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \]
\[ \leq |\mathbf{a}|^2 + 2|\mathbf{a}| |\mathbf{b}| + |\mathbf{b}|^2 \]
\[ \leq (|\mathbf{a}| + |\mathbf{b}|)^2 \]

The result follows.