Lab 5

1. $f(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$

$\forall f \in \mathbb{P}_n$

Choose $f(x) \equiv 1$ to get

$$1 = \sum_{i=0}^{n} 1 \cdot l_i(x)$$

$$= \sum_{i=0}^{n} l_i(x), \quad \forall x \in [a,b]$$

2. $p(x)$ interpolates f at x_i's

$$\Rightarrow P(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x) \quad — ①$$

By #1,

$$f(x) = f(x)\cdot 1 = f(x) \sum_{i=0}^{n} l_i(x)$$

$$= \sum_{i=0}^{n} f(x)\, l_i(x) \quad — ②$$

② − ① :

$$f(x) - P(x) = \sum_{i=0}^{n} \left(f(x) - f(x_i)\right) l_i(x)$$

3. (i) Procedure in sec 6.1 (P311)

- # of addition/subtraction

$: \; 2+5+8+\cdots+3k-1+\cdots+3n-1$

$= 3\cdot\frac{n(n+1)}{2} - n = \frac{3n^2+n}{2} = \frac{n(3n+1)}{2}$

- # of multiplication/division

$: \; 1+3+5+\cdots+2k-1+\cdots+2n-1$

$= n^2$

(ii) Divided Difference

- # of Addition/subtraction

$: \; 2n + 2(n-1) + \cdots + 2\cdot 1 = 2\sum_{k=1}^{n}$

$= 2\,\frac{n(n+1)}{2} = n(n+1)$

- # of multiplication/division

$: \; n + (n-1) + (n-2) + \cdots + 1$

$= \frac{n(n+1)}{2}$

$\therefore$ Divided difference scheme is more efficient.

4. (Do it only for $n=1$ case)

$$\begin{array}{cc|ccc}
x_0 & y_0 & 0 & \frac{y_1-y_0}{(x_1-x_0)^2} & -\frac{2(y_1-y_0)}{(x_1-x_0)^3} \\
x_0 & y_0 & \frac{y_1-y_0}{x_1-x_0} & -\frac{y_1-y_0}{(x_1-x_0)^2} & \\
x_1 & y_1 & 0 & & \\
x_1 & y_1 & & &
\end{array}$$

$$\therefore P(x) = y_0 + \frac{y_1-y_0}{(x_1-x_0)^2}(x-x_0)^2 - \frac{2(y_1-y_0)}{(x_1-x_0)^3}(x-x_0)^2(x-x_1)$$

5. (i) From $f(x) = 3 + x - 9x^2$ on $[0,1]$

$f'(x) = 1 - 18x$, $f''(x) = -18$

$f(1) = -5$, $f'(1) = -17$, $f''(1) = -18$

From $f(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3$ on $[1,2]$

$f'(x) = b + 2c(x-1) + 3d(x-1)^2$

$f''(x) = 2c + 6d(x-1)$

$-5 = f(1) = a \Rightarrow a = -5$

$-17 = f'(1) = b \Rightarrow b = -17$

$-18 = f''(1) = 2c \Rightarrow c = -9$

$\therefore f(x) = -5 - 17(x-1) - 9(x-1)^2 + d(x-1)^3$ on $[1,2]$

To minimize $\int_0^2 [f''(x)]^2\,dx$, minimize

$$\int_1^2 [f''(x)]^2\,dx = \int_1^2 [-18 + 6d(x-1)]^2\,dx$$

$$= 6^2 \int_1^2 [d(x-1)-3]^2\,dx = \frac{36}{3\cdot d}\Big[(d(x-1)-3)^3\Big]_1^2$$

$$= \frac{12}{d}\Big[(d-3)^3 - (-3)^3\Big] = \frac{12}{d}\Big[(d-3)^3 + 3^3\Big]$$

$$= \frac{12}{\cancel{d}}\left\{\cancel{(d-3)+3}\right\}\left\{(d-3)^2 - 3(d-3) + 3^2\right\}$$

$$= 12\left[\left\{(d-3) - \tfrac{3}{2}\right\}^2 - \tfrac{9}{4} + 9\right]$$

$$= 12\left[\left(d - \tfrac{9}{2}\right)^2 + \tfrac{27}{4}\right] = 12\left(d - \tfrac{9}{2}\right)^2 + 81$$

It is minimum when $d = \frac{9}{2}$

(ii) $0 = f''(2) = -18 + 6d \Rightarrow d = 3$

Since $f''(0) = -18 \neq 0$, f is <u>not</u> a natural cubic spline. That is why there is a discrepancy in (i) & (ii)

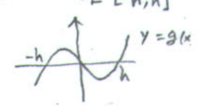
6. $f(x) - P_2(x) = \frac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$

$\therefore \|f - P_2\|_\infty \le \frac{M_3}{6}\|(x-x_0)(x-x_1)(x-x_2)\|_\infty$

Since $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, by translation

$\|(x-x_0)(x-x_1)(x-x_2)\|_{L^\infty[x_0,x_2]} = \|(x+h)x(x-h)\|_{L^\infty[-h,h]}$

Set $g(x) = x(x^2 - h^2) = x^3 - h^2 x$



$g'(x) = 3x^2 - h^2 = 0$

$\Rightarrow x = \pm\frac{h}{\sqrt{3}}$

$\|(x+h)x(x-h)\|_{L^\infty[-h,h]} = g\left(-\frac{h}{\sqrt{3}}\right) = -\frac{h^3}{3\sqrt{3}} + \frac{h^3}{\sqrt{3}}$

$= \frac{2h^3}{3\sqrt{3}}$

$\therefore \|f - P_2\|_\infty \le \frac{M_3}{6}\cdot\frac{2h^3}{3\sqrt{3}} = \frac{M_3}{9\sqrt{3}}h^3$

7. Let $f(x) \in S_m^m$ be given

For given $i = 1, 2, \cdots, n-1$, on $[x_{i-1}, x_i]$,

$f(x) = p_i(x) = c_0 + c_1(x-x_i) + c_2(x-x_i)^2 + \cdots + c_m(x-x_i)^m$

and on $[x_i, x_{i+1}]$,

$f(x) = q_i(x) = d_0 + d_1(x-x_i) + d_2(x-x_i)^2 + \cdots + d_m(x-x_i)^m$

Since $f \in C^0$, $p_i(x_i) = q_i(x_i) \Rightarrow c_0 = d_0$

$f \in C^1 \Rightarrow p_i'(x_i) = q_i'(x_i) \Rightarrow c_1 = d_1$

$f \in C^2 \Rightarrow p_i''(x_i) = q_i''(x_i) \Rightarrow c_2 = d_2$

$\vdots$

$f \in C^m \Rightarrow p_i^{(m)}(x_i) = q_i^{(m)}(x_i) \Rightarrow c_m = d_m$

$\therefore f \in \mathbb{P}_m[x_{i-1}, x_{i+1}]$

Do the same for $i = 1, 2, \cdots, n-1$, to get

$f \in \mathbb{P}_m[x_0, x_n]$