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Lab 4

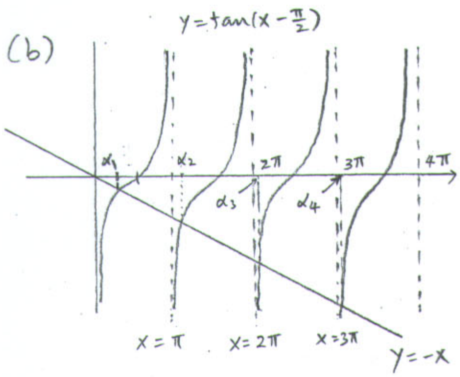
1. (a) $\alpha_1 \approx 0.8603$

$\alpha_2 \approx 3.4256$

$\alpha_3 \approx 6.4373$

$\alpha_4 \approx 9.5293$

(b)



$\therefore \ \alpha_n \sim (n-1)\pi$, as $n \to \infty$

$\therefore \ \lim\limits_{n\to\infty} \dfrac{\alpha_n}{n} \underset{\uparrow}{=} \lim\limits_{n\to\infty} \dfrac{(n-1)\pi}{n} = \pi$

$(\because \alpha_n \sim (n-1)\pi)$

2. $f(x) = x^3 - 3x^2 + x + 3$

(a) $f(-1) = -1 - 3 - 1 + 3 = -2 < 0$

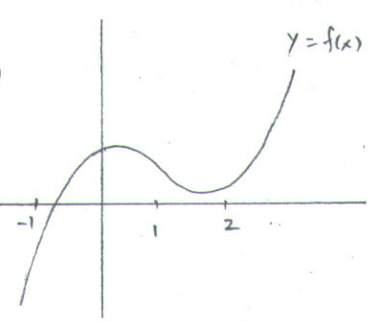
$f(2) = 8 - 12 + 2 + 3 = 1 > 0$

By IVT, $\exists\, \alpha \in (-1, 2)$ s.t. $f(\alpha) = 0$

(b) $x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$

$$\left(\begin{aligned} &\text{(i)} \quad 1 - \frac{f(1)}{f'(1)} = 1 - \frac{2}{-2} = 2 \\ &\phantom{\text{(i)}} \quad 2 - \frac{f(2)}{f'(2)} = 2 - \frac{1}{1} = 1 \end{aligned} \right)$$

(c)



$x_0 = -1$ is a good choice

$\alpha \approx -0.76929235$

3. (a) $\left[\begin{array}{l} x_n \to 0, \text{ as } n \to \infty \\ \text{if } x_0 = 0.5, 1, 1.3 \end{array} \right.$

$\left[\begin{array}{l} x_n \to \infty, \text{ as } n \to \infty \\ \text{if } x_0 = 1.4, 2, 3 \end{array} \right.$

(b) If $0 < x_0 \leq 1.391$

then $x_n \to 0$, as $n \to \infty$

(c) $f(x) = \tan^{-1}x$, $f'(x) = \dfrac{1}{1+x^2}$

$x_{n+1} = x_n - \dfrac{\tan^{-1}x_n}{\frac{1}{1+x_n^2}} = x_n - (1+x_n^2)\tan^{-1}x_n$

(Case 1 : $x_n > 0$)

$x_{n+1} < 0 \iff x_n - (1+x_n^2)\tan^{-1}x_n < 0$

$\iff \tan^{-1}(x_n) > \dfrac{x_n}{1+x_n^2}$

$\iff \dfrac{\tan^{-1}(x_n)}{x_n} > \dfrac{1}{1+x_n^2}$ — ①

Note: $\dfrac{\tan^{-1}(x_n)}{x_n} = \dfrac{\tan^{-1}(x_n) - \tan^{-1}(0)}{x_n - 0} = \dfrac{1}{1+\xi_n^2}$

for some $\xi_n \in (0, x_n)$ by MVT

Since $\dfrac{1}{1+\xi_n^2} > \dfrac{1}{1+x_n^2}$, ① holds

$\therefore \ x_{n+1} < 0$, if $x_n > 0$

(case 2 : $x_n < 0$)

By similar argument, $x_{n+1} > 0$

By Case 1 & Case 2, if $x_0 \neq 0$,

$x_{n+1} \cdot x_n < 0$, $\forall n \geq 0$

(d) (Case 1 : $x_n > 0$)

$|x_{n+1}| < |x_n| \iff -x_{n+1} < x_n$

$\iff -x_n + (1+x_n^2)\tan^{-1}x_n < x_n$

$\iff \tan^{-1}x_n < \dfrac{2x_n}{1+x_n^2}$

$\iff \dfrac{2x_n}{1+x_n^2} - \tan^{-1}x_n > 0$ — ②

(Case 2 : $x_n < 0$)

$|x_{n+1}| < |x_n| \iff x_{n+1} < -x_n$

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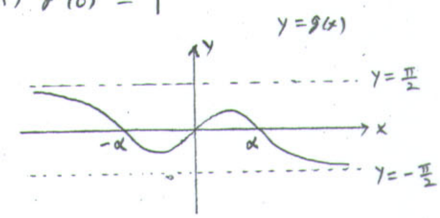
$\iff \dfrac{2x_n}{1+x_n^2} - \tan^{-1}x_n < 0$ — ③

(e) set $g(x) = \dfrac{2x}{1+x^2} - \tan^{-1}x$

$$\left\{ \begin{array}{l} \text{(i) } g \text{ is an odd ftn} \\ \text{(ii) } \lim\limits_{x\to\infty} g(x) = -\frac{\pi}{2} \\ \qquad \lim\limits_{x\to-\infty} g(x) = \frac{\pi}{2} \\ \text{(iii) } g'(0) = 1 \end{array} \right.$$



$g(\alpha) = 0$, $\alpha > 0 \Rightarrow \alpha = 1.3917452002\cdots$

② $\iff g(x_n) > 0$ (if $x_n > 0$)

③ $\iff g(x_n) < 0$ (if $x_n < 0$)

$\therefore$ If $|x_n| < \alpha$ then $|x_{n+1}| < |x_n|$.

$\therefore$ Newton's Method converges,

if $|x_0| < \alpha = 1.3917452002\cdots$