Lab 3

1. $\frac{1}{9} = (0.\overline{000111})_2$

$= (1.11\overline{000111})_2 \times 2^{-4}$

$= (1.11000111\cdots\cdots 00011100\cdots)_2 \times 2^{-4}$ (↑43 ↑44)

$\therefore x_+ = (1.11000111000111\cdots 100100)_2 \times 2^{-4}$ (↑43)

2. Suppose $x > 0$

$x = \underbrace{(0.a_1a_2\cdots a_{12}\cdots)_{10}}_{\equiv r} \times 10^n$

where $0 \le a_i \le 9$, $a_1 \ne 0$

$x_- = (0.a_1a_2\cdots a_{12}) \times 10^n$

$x_+ = [(0.a_1a_2\cdots a_{12}) + 10^{-12}] \times 10^n$

$|x - fl(x)| \le \frac{1}{2}|x_+ - x_-| = \frac{1}{2}10^{-12}\cdot 10^n$

$\frac{|x-fl(x)|}{|x|} \le \frac{\frac{1}{2}10^{n-12}}{r\times 10^n} \le \frac{\frac{1}{2}10^{-12}}{\frac{1}{10}}$

$= \frac{1}{2}10^{-11}$

$fl(x) = x(1+\delta)$, $|\delta| \le \varepsilon = \frac{1}{2}\cdot 10^{-11}$

3. $x = 1$, $y = \cos(\frac{1}{4})$. Then,

$1 - \frac{\cos(\frac{1}{4})}{1} \approx 0.0310875783$

$2^{-5} = 0.03125$

$2^{-6} = 0.015625$

$\therefore 2^{-6} \le 1 - \frac{\cos(\frac{1}{4})}{1} < 2^{-5}$

$\therefore$ 5 or 6 significant binary digits are lost.

4. $f(x) = \frac{1-\cos x}{x}$

(a) $\lim_{x\to 0}\frac{1-\cos x}{x} \overset{H}{=} \lim_{x\to 0}\frac{\sin x}{1} = 0$

$\therefore f(0) = 0$

(b) $x \approx 2n\pi$, $n \in \mathbb{Z}$

(c) $\frac{1-\cos x}{x} = \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)}$

$= \frac{1-\cos^2 x}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)}$

$\therefore$ If $x \approx 2n\pi$, use

$f(x) = \frac{\sin^2 x}{x(1+\cos x)}$ — ①

(d) If $x \approx (2n+1)\pi$, $n \in \mathbb{Z}$,

then ① has cancellation error in $1+\cos x$, since $\cos x \approx -1$

In this case, use the original form

$f(x) = \frac{1-\cos x}{x}$

5. $|r - c_n| \le 2^{-(n+1)}(b_0 - a_0)$

$\therefore \frac{|r-c_n|}{|r|} \le \frac{2^{-(n+1)}(b_0-a_0)}{a_0}$

$(\because a_0 \le r \le b_0)$

$\frac{2^{-(n+1)}(b_0-a_0)}{a_0} \le \varepsilon$

$\Leftrightarrow 2^{-(n+1)} \le \frac{\varepsilon a_0}{(b_0-a_0)}$

$\Leftrightarrow -(n+1)\ln 2 \le \ln\varepsilon + \ln a_0 - \ln(b_0-a_0)$

$n+1 \ge \frac{\ln(b_0-a_0) - \ln\varepsilon - \ln a_0}{\ln 2}$

$\therefore n \ge \frac{\ln(b_0-a_0) - \ln\varepsilon - \ln a_0}{\ln 2} - 1$

6. $f(x) = x^{\frac{1}{3}}$ $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$\therefore x_{n+1} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3}x_n^{-\frac{2}{3}}} = x_n - 3x_n$

$= -2x_n$

$\therefore x_n = (-2)^n x_0$

$\Rightarrow$ diverges for $\forall x_0 \ne 0$

Note that Thm 1 in sec 3.2 does not apply to this ftn, because f is not C^2 around 0 ($\because \nexists f'(0)$)