305 # Lab 2 S09

1. $x_{n+2} - x_{n+1} = F(x_{n+1}) - F(x_n)$

$= F'(\xi_n)(x_{n+1} - x_n)$, $\underbrace{\xi_n \in I(x_{n+1} - x_n)}_{①}$

Since $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} \xi_n = x$

Since F is C^1 (i.e., F' is conti),

$\lim_{n\to\infty} F'(\xi_n) = F'(x) = 0$ — ②

By ① & ②,

$x_{n+2} - x_{n+1} = o(x_{n+1} - x_n)$

2. (a) $\Rightarrow$ (b)

(Pf) Suppose that (a) holds, i.e.,

$\exists C>0,\ \varepsilon>0,\ K>0$ s.t.

$|f(x)| \le C|x|^{-n-\varepsilon}$, $\forall |x| \ge K$

$\therefore \dfrac{|f(x)|}{|x|^{-n}} \le C|x|^{-\varepsilon} = \dfrac{C}{|x|^{\varepsilon}}$, $\forall |x| \ge K$

Since $\lim_{|x|\to\infty} \dfrac{C}{|x|^{\varepsilon}} = 0$, $\lim_{|x|\to\infty} \dfrac{|f(x)|}{|x|^{-n}} = 0$

$\therefore |f(x)| = o(|x|^{-n})$

(b) $\not\Rightarrow$ (a)

Consider $f(x) = \dfrac{1}{|x|^n \ln|x|}$

Clearly $\lim_{|x|\to\infty} \dfrac{|f(x)|}{|x|^{-n}} = \lim_{|x|\to\infty} \dfrac{1}{\ln|x|} = 0$

$\therefore f(x) = o(|x|^{-n})$

However, $\forall \varepsilon>0, C>0, K>0, \exists |x| \ge K$

s.t. $\dfrac{1}{\ln|x|} > \dfrac{C}{|x|^{\varepsilon}}$

$(\Leftrightarrow |f(x)| > C|x|^{-n-\varepsilon})$

$\therefore \forall \varepsilon > 0$, $|f(x)| \ne O(|x|^{-n-\varepsilon})$

3. $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$

Lagrange Remainder Thm

$= \sum_{k=0}^{n} x^k + \dfrac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$, where

$f(x) = (1-x)^{-1}$, $f'(x) = (-1)(1-x)^{-2}$, ...

$f^{(n+1)}(x) = (n+1)!\,(-1)^{n+1}(1-x)^{-(n+1)}$

$\therefore \dfrac{1}{1-x} = \sum_{k=0}^{n} x^k + \dfrac{(-1)^{n+1}}{(1-\xi)^{n+1}} x^{n+1}$,

$\therefore \sum_{k=0}^{n} x^k = \dfrac{1}{1-x} + \dfrac{(-1)^n}{(1-\xi)^{n+1}} x^{n+1}$,

where $\xi \in I(0,x)$

as $x \to 0$, $\xi \to 0$

$\therefore \left| \sum_{k=0}^{n} x^k - \dfrac{1}{1-x} \right| \le \dfrac{|x|^{n+1}}{|1-\xi|^{n+1}}$

$\therefore \dfrac{\left| \sum_{k=0}^{n} x^k - \frac{1}{1-x} \right|}{|x|^n} \le \dfrac{|x|}{|1-\xi|^{n+1}} \to 0$,

as $x \to 0$

$\therefore \sum_{k=0}^{n} x^k = \dfrac{1}{1-x} + o(|x|^n)$

4. (a) $e^x = 1 + x + \dfrac{e^{\xi}}{2} x^2$, $\xi \in I(0,x)$

$e^x - 1 = x + \dfrac{e^{\xi}}{2} x^2$

$\ne O(x^2)$, as $x \to 0$

(b) $\left[\begin{array}{l} \cot x = o(x^{-1}),\ \text{as } x \to 0 \\ \Leftrightarrow \lim_{x\to 0} \dfrac{\cot x}{x^{-1}} = 0 \end{array} \right.$

However, $\lim_{x\to 0} \dfrac{\cot x}{x^{-1}} = \lim_{x\to 0} \dfrac{x}{\tan x}$

$\overset{(H)}{=} \lim_{x\to 0} \dfrac{1}{\sec^2 x} = 1$

(c) Suppose that $x^{-2} = O(\cot x)$, as $x \to 0$

i.e., $\exists C>0, \delta>0$, s.t.

$x^{-2} \le C|\cot x|$, $\forall\, 0<|x|<\delta$

$\therefore 0 < \dfrac{1}{C} \le \dfrac{|\cot x|}{x^{-2}}$, $\forall\, 0<|x|<\delta$ — ①

However, $\lim_{x\to 0} \dfrac{\cot x}{x^{-2}} = \lim_{x\to 0} \dfrac{x^2}{\tan x}$

$= \lim_{x\to 0} \dfrac{2x}{\sec^2 x} = 0$ $\therefore \rightarrow\!\leftarrow$ ①

5. (a) $e^h = 1 + h + \dfrac{e^{\xi}}{2} h^2$, $\xi \in I(0,h)$

$\therefore e^h = 1 + O(h)$, $e^h = 1 + o(1)$

$\underline{\alpha = 1,\ \beta = 0}$

(b) $\cosh = 1 + \dfrac{h^2}{2} + \dfrac{\cos\xi}{4!} h^4$

$\cosh = 1 + O(h^2)$, $\cosh = 1 + o(h)$

$\underline{\alpha = 2,\ \beta = 1}$

(c) $1 + \sin(h^3) = 1 + h^3 - \dfrac{\cos\xi}{6} h^6$

$\therefore 1 + \sin(h^3) = 1 + O(h^3)$, $1 + o(h^2)$

$\therefore \underline{\alpha = 3,\ \beta = 2}$

(d) $\dfrac{1}{1-h^4} = 1 + h^4 + (h^4)^2 + (h^4)^3 + \cdots$

$\therefore \dfrac{1}{1-h^4} = 1 + O(h^4)$, $1 + o(h^3)$

$\therefore \underline{\alpha = 4,\ \beta = 3}$

6. (a) $a_n \to 5$, as $n \to \infty$

(In fact, $\underline{a_n = 5 - (\tfrac{1}{2})^{n-3}}$, $\underline{\forall n \ge 2}$

(∵) $a_{n+2} = \frac{3}{2} a_{n+1} - \frac{1}{2} a_n$

$\Rightarrow a_{n+2} - a_{n+1} = \frac{1}{2}(a_{n+1} - a_n)$

$\therefore \left[\begin{array}{l} b_{n+1} = \frac{1}{2} b_n \\ b_1 = a_2 - a_1 = 2 \end{array} \right.$ $(\underbrace{b_k \equiv a_{k+1} - a_k})$

$\Rightarrow b_k = 2^{2-k}$

$a_n = a_1 + b_1 + b_2 + b_3 + \cdots + b_{n-1}$

$= a_1 + \sum_{k=1}^{n-1} b_k = 1 + \sum_{k=1}^{n-1} 2^{2-k}$

$= 1 + \dfrac{2\{1 - (\frac{1}{2})^{n-1}\}}{1 - \frac{1}{2}} = 1 + 4(1 - (\tfrac{1}{2})^{n-1})$

$= 5 - (\tfrac{1}{2})^{n-3}$)

(b) order of conv = 1 (i.e., linear conv)

$\left(\lim_{n\to\infty} \dfrac{|a_{n+1} - 5|}{|a_n - 5|} = \lim_{n\to\infty} \dfrac{(\frac{1}{2})^{n-2}}{(\frac{1}{2})^{n-3}} = \dfrac{1}{2} \right)$

7. (a) $x_n \to \sqrt{3}$, as $n \to \infty$

(If you assume that $\lim_{n\to\infty} x_n = \alpha > 0$

$x_{n+1} = \frac{1}{2} x_n + \dfrac{3}{2x_n}$

$\downarrow \quad \downarrow \quad \downarrow$ as $n \to \infty$

$\alpha = \frac{1}{2}\alpha + \dfrac{3}{2\alpha}$

$\therefore \dfrac{\alpha}{2} = \dfrac{3}{2\alpha} \Rightarrow \alpha^2 = 3$

$\therefore \alpha = \sqrt{3}$ $(\because \alpha > 0)$)

(b) order of conv = 2

(In fact, $\{x_n\}$ is obtained from Newton's method to the equation $\underbrace{x^2 - 3}_{=f(x)} = 0$

See chapter 3.)