305 — Lab 1 — S09

1. (#10 on P13)

(i) If $\alpha = 0$, then obvious

(ii) Suppose $\alpha \neq 0$. Then,

$$\frac{f(x+h)-f(x+\alpha h)}{h-\alpha h}$$

$$= \frac{f(x+h)-f(x)+f(x)-f(x+\alpha h)}{(1-\alpha)h}$$

$$= \frac{1}{1-\alpha}\,\frac{f(x+h)-f(x)}{h} - \frac{\alpha}{1-\alpha}\cdot\frac{f(x+\alpha h)-f(x)}{\alpha h}$$

$$\to \frac{1}{1-\alpha}f'(x) - \frac{\alpha}{1-\alpha}f'(x), \text{ as } h\to 0$$

$$= \frac{1-\alpha}{1-\alpha}f'(x) = f'(x)$$

2. (#30 on P14)

$y = x^x \to \ln y = x\ln x$

$\therefore \frac{y'}{y} = \ln x + x\cdot\frac{1}{x} = \ln x + 1$

$\therefore y' = y(\ln x+1) = x^x(\ln x+1)$

$y'' = y'(\ln x+1) + y(\ln x+1)'$

$= x^x(\ln x+1)^2 + x^x\left(\frac{1}{x}\right)$

$= x^x\left[(\ln x+1)^2 + \frac{1}{x}\right]$

$y(1) = 1,\ y'(1) = 1$

$\therefore T_1(x) = 1 + 1\cdot(x-1) = x$

$E_1(x) = \frac{1}{2}f''(\xi)(x-1)^2 = \frac{1}{2}\xi^\xi\left[(\ln\xi+1)^2+\frac{1}{\xi}\right]\cdot(x-1)^2$

for some $\xi \in I(1,x)$

3. (#33 on P14)

(a) Use Lagrange Remainder's Thm with $n=3$ to get

$$\cos x = 1 - \frac{x^2}{2} + \frac{\cos\xi}{4!}x^4$$

for some $\xi \in I(0,x)$.

Since $|x| < \frac{1}{2}$,

$$\left|\cos x - \left(1-\frac{x^2}{2}\right)\right| = \frac{|\cos\xi|}{24}|x|^4 < \frac{1}{24}\left(\frac{1}{2}\right)^4 = \boxed{\frac{1}{384}}$$

(b) Use LR Thm with $n=4$

$$\sin x = x - \frac{x^3}{6} + \frac{\cos\xi}{5!}x^5$$

$$\therefore \left|\sin x - x\left(1-\frac{x^2}{6}\right)\right| = \frac{|\cos\xi|}{5!}|x|^5$$

Since $\xi \in I(0,x)$ & $|x| < \frac{1}{2}$

$|\xi| < \frac{1}{2}$ so that $|\cos\xi| \le 1$

$$\therefore \left|\sin x - x\left(1-\frac{x^2}{6}\right)\right| < \frac{1}{120}\left(\frac{1}{2}\right)^5 = \frac{1}{3840}$$

4. $f^{(k)}(a) = g^{(k)}(a),\ \forall\, 0 \le k \le n$ — ①

$$f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$$g(x) = \sum_{k=0}^{n}\frac{g^{(k)}(a)}{k!}(x-a)^k + \frac{g^{(n+1)}(d)}{(n+1)!}(x-a)^{n+1}$$

for some $c, d \in I(a,x)$.

$$① \Rightarrow f(x)-g(x) = \frac{f^{(n+1)}(c)-g^{(n+1)}(d)}{(n+1)!}(x-a)^{n+1}$$

$$\therefore \frac{f(x)-g(x)}{(x-a)^n} = \frac{f^{(n+1)}(c)-g^{(n+1)}(d)}{(n+1)!}(x-a)$$

Since $c, d \in I(a,x)$, $\lim_{x\to a} c = \lim_{x\to a} d = a$

$\therefore \lim_{x\to a} f^{(n+1)}(c) = f^{(n+1)}(a)$, $\lim_{x\to a} g^{(n+1)}(d) = g^{(n+1)}(a)$

$$\therefore \lim_{x\to a}\frac{f(x)-g(x)}{(x-a)^n} = 0$$

5. By Cauchy Remainder Thm,

$$E_n(x) = \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt \quad — ①$$

(i) Either $(x-t)^n \ge 0,\ \forall t \in I(x_0,x)$
or $(x-t)^n \le 0,\ \forall t \in I(x_0,x)$

Set $u(t) = f^{(n+1)}(t)$ & $v(t) = (x-t)^n$

By (i), $f \in C^{n+1}(I)$, & MVT for integrals

$$\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt$$

$$= f^{(n+1)}(\xi)\int_{x_0}^{x}(x-t)^n\,dt$$

$$= \frac{-1}{n+1}f^{(n+1)}(\xi)\left[(x-t)^{n+1}\right]_{t=x_0}^{t=x}$$

$$= \frac{f^{(n+1)}(\xi)}{n+1}(x-x_0)^{n+1}, \text{ for some } \xi \in I(x_0,x) \quad — ②$$

① & ② implies

$$E_n(x) = \frac{1}{(n+1)\,n!}f^{(n+1)}(\xi)(x-x_0)^{n+1}$$

$$= \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

$\therefore$ Lagrange Remainder Thm holds.

6. (a) $f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k},\ \forall k = 1, 2, \cdots$

$\therefore f(0) = 0,\ f^{(k)}(0) = (-1)^{k-1}(k-1)!,\ \forall k \ge 1$

$$\therefore P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}x^k$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}}{n}x^n$$

(b) (i) $x \in [0,1]$: $\exists c \in I(0,x)$ s.t.

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{(-1)^n x^{n+1}}{(n+1)(1+c)^{n+1}}. \quad \left(\text{Since } 1+c>1, \ \& \ 0 \le x \le 1,\right)$$

$$|R_{n+1}(x)| = \frac{x^{n+1}}{(n+1)(1+c)^{n+1}} \le \frac{1}{n+1},\ \forall n,\ \forall x \in [0,1]$$

$$\therefore \lim_{n\to\infty} R_{n+1}(x) = 0,\ \forall x \in [0,1]$$

(ii) $-\frac{1}{2} \le x < 0 \Rightarrow -\frac{1}{2} < c < 0 \Rightarrow \frac{1}{2} < 1+c < 1$

$$\therefore \frac{|x|}{|1+c|} < \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$\therefore |R_{n+1}(x)| \le \frac{1}{n+1},\ \forall n,\ \forall x \in [-\frac{1}{2}, 0)$

$\therefore \lim_{n\to\infty} R_{n+1}(x) = 0,\ \forall x \in [-\frac{1}{2}, 0)$

(iii) $-1 < x < -\frac{1}{2}$, $c \in I(0,x)$

$\therefore c$ could be in $(-1, -\frac{1}{2})$. In that case $\frac{|x|}{1+c} > \frac{\frac{1}{2}}{\frac{1}{2}} = 1$ might happen!

Then, the above approach does not work.

(c) $R_{n+1} = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt$

$$= \int_0^x \frac{(-1)^n}{(1+t)^{n+1}}(x-t)^n\,dt = \int_0^x \frac{(t-x)^n}{(1+t)^{1+n}}\,dt$$

(d) $-1 < x \le t \le 0 \Rightarrow 1+t > 0,\ t-x \ge 0 \Rightarrow \frac{t-x}{1+t} \ge 0$

And $(-xt-x) - (t-x) = -xt - t = -t(1+x) \ge 0$

$$\therefore 0 \le \frac{t-x}{1+t} \le \frac{-xt-x}{1+t} = \frac{-x(1+t)}{1+t} = -x = |x|$$

(e) $|R_{n+1}(x)| \le \int_x^0 \left|\frac{t-x}{1+t}\right|^n \frac{1}{1+t}\,dt$ (by (c))

$$\overset{(d)}{\le} |x|^n\int_x^0 \frac{1}{1+t}\,dt = |x|^n\left[\ln(1+t)\right]_x^0 = -|x|^n\ln(1+x)$$

(Note that $0 < 1+x < 1 \Rightarrow \ln(1+x) < 0$)

(f) Since $-1 < x < 0$,

$$\lim_{n\to\infty}|x|^n = 0$$

$$\therefore \lim_{n\to\infty}\left(-|x|^n\ln(1+x)\right) = 0$$

$\therefore$ By this & (e)

$$\lim_{n\to\infty}|R_{n+1}(x)| = 0,\ \forall x \in (-1,0)$$

$$\therefore \lim_{n\to\infty}P_n(x) = \ln(1+x),\ \forall x \in (-1,0)$$