4. Done in class

7. (a) Let $w(x) = x$ on $[0,1]$ & $n=1$. Then,

$\int_0^1 x f(x)\,dx = \int_0^1 f(x)\,w(x)\,dx \approx A_0 f(x_0) + A_1 (x_1)$

is exact for $\forall f \in \mathbb{P}_3$, if x_0, x_1 are the root of $q(x)$ which satisfy

(i) $q \in \mathbb{P}_2$

(ii) $q \perp_w \mathbb{P}_1$

$\left(\text{i.e., } \int_0^1 q(x) f(x)\, x\,dx = 0, \ \forall f \in \mathbb{P}_1 \right)$

Find such q applying Gram Schmidt process to $\{1, x, x^2\}$

$q_0(x) \equiv 1 \qquad \|q_0\|^2 = \int_0^1 1^2 \cdot x\,dx = \frac{1}{2}$

$q_1(x) = x - \frac{\langle x, 1\rangle}{\|1\|^2}\cdot 1 = x - 2\int_0^1 x\cdot x\,dx$

$= x - \frac{2}{3}$

$\|q_1\|^2 = \int_0^1 (x-\frac{2}{3})^2 x\,dx = \cdots = \frac{1}{36}$

$q_2(x) = x^2 - \frac{\langle x^2, 1\rangle}{\|1\|^2}\cdot 1 - \frac{\langle x^2, x-\frac{2}{3}\rangle}{\|x-\frac{2}{3}\|^2}(x-\frac{2}{3})$

$= x^2 - 2\int_0^1 x^3\,dx - 36\int_0^1 x^2(x-\frac{2}{3})x\,dx\cdot(x-\frac{2}{3})$

$= x^2 - \frac{6}{5}x + \frac{3}{10}$

$q_2(x) = 0 \Rightarrow \therefore x_0 = \frac{6-\sqrt{6}}{10} \qquad x_1 = \frac{6+\sqrt{6}}{10}$

$\therefore \int_0^1 x f(x)\,dx = A_0 f\left(\frac{6-\sqrt{6}}{10}\right) + A_1 f\left(\frac{6+\sqrt{6}}{10}\right)$

$$\left.\begin{array}{l} f(x)=1 \rightarrow \frac{1}{2} = A_0 + A_1 \\ f(x)=x \rightarrow \frac{1}{3} = \frac{6-\sqrt{6}}{10}A_0 + \frac{6+\sqrt{6}}{10}A_1 \end{array}\right\} \Rightarrow \begin{array}{l} A_0 = \frac{9-\sqrt{6}}{36} \\ A_1 = \frac{9+\sqrt{6}}{36} \end{array}$$

$\therefore \int_0^1 x f(x)\,dx = \underline{\frac{9-\sqrt{6}}{36} f\left(\frac{6-\sqrt{6}}{10}\right) + \frac{9+\sqrt{6}}{36} f\left(\frac{6+\sqrt{6}}{10}\right)}$

11. We know $\int_{-1}^1 f(t)\,dt \approx f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$ is exact on $\mathbb{P}_3$.

$\int_0^2 f(x)\,dx \underset{x=t+1}{=} \int_{-1}^1 f(t+1)\,dt \underset{g(t)=f(t+1)}{\equiv} \int_{-1}^1 g(t)\,dt \approx g(-\frac{1}{\sqrt{3}}) + g(\frac{1}{\sqrt{3}})$

$= f(1-\frac{1}{\sqrt{3}}) + f(1+\frac{1}{\sqrt{3}}) = f(1-\frac{1}{\sqrt{3}}) + f[2-(1-\frac{1}{\sqrt{3}})]$

$\therefore \alpha = 1 - \frac{1}{\sqrt{3}}$

14. For arbitrarily given $p \in \mathbb{P}_n$, (let $q_{n+1}(x) = \prod_{j=0}^{n}(x-x_j)$)

$\int_a^b \underbrace{q_{n+1}(x)\,p(x)}_{\in \mathbb{P}_{2n+1}} w(x)\,dx = \sum_{i=0}^{n} A_i \underbrace{q_{n+1}(x_i)}_{=0,\ \forall i} p(x_i) = 0$

$\therefore q_{n+1} \perp_w p, \ \forall p \in \mathbb{P}_n \qquad \therefore q_{n+1} \perp_w \mathbb{P}_n$

18. Let $w(x) \equiv (x^4-1)$ on $[1,2]$. Then,

$$\begin{cases} \int_1^2 (x^4-1) f(x)\,dx = \int_1^2 f(x) w(x)\,dx \approx A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2) \\ \text{is correct for all } f \in \mathbb{P}_5 = \mathbb{P}_{2\cdot 2+1} \end{cases}$$

$\Rightarrow (x-x_0)(x-x_1)(x-x_2) \perp \mathbb{P}_2$ w.r.t x^4-1

If $q \in \mathbb{P}_3$ & $q \perp_w \mathbb{P}_2$, then x_0, x_1, x_2 are the roots of q.

19. $\int_0^1 x^i (35x^4 - 60x^2 + 32x - 3)\,dx = 0 \quad \forall i = 0,1,2$

$\therefore 35x^4 - 60x^2 + 32x - 3 \perp \mathbb{P}_2$ wrt $w(x)=1$ on $[0,1]$

(Note: none of the other polynomial is $\perp$ to $\mathbb{P}_2$ wrt $w(x)=1$ on $[0,1]$)