HW 7.2

1. $\int_0^1 f(x)dx$

$$\approx \frac{1}{8}f(0)+\frac{3}{8}f(\tfrac{1}{3})+\frac{3}{8}f(\tfrac{2}{3})+\frac{1}{8}f(1)$$

2. (i) $f(x)=1$

RHS $= \frac{b-a}{6}[1+4+1] = b-a = \int_a^b 1\,dx$ ✓

(ii) $f(x)=x$

RHS $= \frac{b-a}{6}[a+2(a+b)+b] = \frac{b-a}{6}3(a+b)$

$= \frac{1}{2}(b^2-a^2) = \int_a^b x\,dx$ ✓

(iii) $f(x)=x^2$

RHS $= \frac{b-a}{6}[a^2+(a^2+2ab+b^2)+b^2]$ ✓

$= \frac{1}{3}(b-a)(a^2+ab+b^2) = \frac{1}{3}(a^3-b^3) = \int_a^b x^2dx$

(iv) $f(x)=x^3$

RHS $= \frac{b-a}{6}[a^3+\frac{1}{2}(a^3+3a^2b+3ab^2+b^3)+b^3]$

$= \frac{b-a}{6}[\frac{3}{2}a^3+\frac{3}{2}a^2b+\frac{3}{2}ab^2+\frac{3}{2}b^3]$

$= \frac{1}{4}(b-a)(a^3+a^2b+ab^2+b^3)$

$= \frac{1}{4}(b^4-a^4) = \int_a^b x^3dx$ ✓

3. $\int_0^1 g(t)dt \approx \frac{1}{6}g(0)+\frac{4}{6}g(\frac{1}{2})+\frac{1}{6}g(1)$ — (5)

$\int_a^b f(x)dx$ $\quad x=a+(b-a)t,\ dx=(b-a)dt$

$= \int_0^1 \underbrace{f(a+(b-a)t)}_{\equiv g(t)}(b-a)dt$

$= (b-a)\int_0^1 g(t)dt$

$\approx \frac{b-a}{6}[g(0)+4g(\frac{1}{2})+g(1)]$

$= \frac{b-a}{6}[f(a)+4f(a+b)+f(b)]$ — (6)

4. (i) $f(x)=1$

RHS $= \frac{1}{90}[7+32+12+32+7] = 1$

$= \int_0^1 1\,dx$ ✓

(ii) $f(x)=x$

RHS $= \frac{1}{90}[\cancel{\frac{32}{4}}^{8}+\cancel{\frac{12}{2}}^{6}+\cancel{32\frac{3}{4}}^{24}+7] = \frac{45}{90}$

$= \frac{1}{2} = \int_0^1 x\,dx$ ✓

Case: $f(x)=x^2, x^3, x^4$ done similarly

6. $\int_0^1 \frac{dt}{t+1}dt$

$= \frac{1}{90}[7\cdot 1+\frac{32}{\frac{1}{4}+1}+\frac{12}{1+\frac{1}{2}}+\frac{32}{1+\frac{3}{4}}+\frac{7}{1+1}]$

$= \frac{1}{90}[7+\frac{128}{1+4}+\frac{24}{2+1}+\frac{128}{4+3}+\frac{7}{2}]$

≈ 0.6931746032

$\ln 2 = 0.6931471806$

$\therefore \ln 2 - \frac{1}{90}[\cdots] = -0.0000274226$

$\approx -2.74\times 10^{-5}$

8. (i) $f(x)=e^x$

$A_0+eA_1 = \int_0^1 e^x dx = e-1$

(ii) $f(x)=\cos(\frac{\pi x}{2})$

$A_0+0 = \int_0^1 \cos\frac{\pi x}{2}dx = \frac{2}{\pi}[\sin\frac{\pi x}{2}]_0^1 = \frac{2}{\pi}$

$eA_1 = e-1-A_0 = e-1-\frac{2}{\pi}$

$\therefore A_1 = 1-\frac{1}{e}-\frac{2}{\pi e}$

$\therefore \int_0^1 f(x)dx \approx \frac{2}{\pi}f(0)+(1-\frac{1}{e}-\frac{2}{\pi e})f(1)$

$\approx (0.6366198)f(0)+(0.39792212)f(1)$

10. $x_0=\frac{1}{3},\quad x_1=\frac{2}{3}$

$\ell_0(x) = \frac{x-\frac{2}{3}}{\frac{1}{3}-\frac{2}{3}} = -3(x-\frac{2}{3}) = -3x+2$

$\ell_1(x) = \frac{x-\frac{1}{3}}{\frac{2}{3}-\frac{1}{3}} = 3(x-\frac{1}{3}) = 3x-1$

$A_0 = \int_0^1 \ell_0(x)dx = \int_0^1(-3x+2)dx = -\frac{3}{2}[x^2]_0^1+2$

$= -\frac{3}{2}+2 = \frac{1}{2}$

$A_1 = \int_0^1 \ell_1(x)dx = \int_0^1(3x-1)dx = \frac{3}{2}[x^2]_0^1-1 = \frac{1}{2}$

$\therefore \int_0^1 f(x)dx = \frac{1}{2}f(\frac{1}{3})+\frac{1}{2}f(\frac{2}{3})$

Use $x=a+(b-a)t$ to get

$\int_a^b f(x)dx = \frac{b-a}{2}[f(\frac{2a+b}{3})+f(\frac{a+2b}{3})]$

11. $P(f;x_1,x_2)(x) = f(x_1)+\frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1) \equiv P(x)$

$\int_{x_0}^{x_3} P(x)dx = f(x_1)(x_3-x_0)+\frac{f(x_2)-f(x_1)}{2(x_2-x_1)}\underbrace{\{(x_3-x_1)^2-(x_0-x_1)^2\}}_{=(x_3-x_0)(x_3-2x_1+x_0)}$

$= (x_3-x_0)[f(x_1)+\frac{x_3-2x_1+x_0}{2(x_2-x_1)}(f(x_2)-f(x_1))]$

$= (x_3-x_0)[\frac{-x_0+2x_2-x_3}{2(x_2-x_1)}f(x_1)+\frac{x_0-2x_1+x_3}{2(x_2-x_1)}f(x_2)]$

13. $f(x)=1 \rightarrow A+B+C = \int_0^2 x\,dx = 2$

$f(x)=x \rightarrow B+2C = \int_0^2 x^2dx = \frac{8}{3}$

$f(x)=x^2 \rightarrow B+4C = \int_0^2 x^3dx = 4$

$\Rightarrow A=0,\ B=\frac{4}{3},\ C=\frac{2}{3}$

$\therefore \int_0^2 xf(x)dx \approx \frac{4}{3}f(1)+\frac{2}{3}f(2)$

If $f(x)=x^3$, then

$\frac{4}{3}f(1)+\frac{2}{3}f(2) = \frac{4}{3}+\frac{16}{3} = \frac{20}{3} \neq \int_0^2 x^4dx = \frac{32}{5}$

$\therefore$ Maximum degree $=2$

31. $f(x)=x+e^{-x^2}$

$f'(x) = 1-2xe^{-x^2}$

$f''(x) = -2e^{-x^2}-2x(-2x)e^{-x^2}$

$= (4x^2-2)e^{-x^2}$

$a=1,\quad b=2\quad h=\frac{2-1}{n}=\frac{1}{n}$

Error $= -\frac{1}{12}(2-1)h^2f''(\xi)$

$\therefore |\text{Error}| \le \frac{\|f''\|_\infty}{12}\frac{1}{n^2}$ — (*)

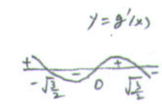
To get $\|f''\|_\infty$, set $g(x)=(4x^2-2)e^{-x^2}$

$g'(x) = 8xe^{-x^2}-2x(4x^2-2)e^{-x^2}$

$= (-8x^3+12x)e^{-x^2}$

$= -4x(2x^2-3)e^{-x^2}$

$g'(x)=0$ at $x=0,\pm\sqrt{\frac{3}{2}}$



$\therefore g(x) = f''(x) = |f''(x)|$ (on $[1,2]$)

attains max value at $\sqrt{\frac{3}{2}}$

$\therefore \|f''\|_\infty = g(\sqrt{\frac{3}{2}}) = (4\frac{3}{2}-2)e^{-\frac{3}{2}}$

$= 4e^{-\frac{3}{2}}$

$\therefore$ (*) $\Rightarrow |\text{Error}| \le \frac{e^{-\frac{3}{2}}}{3}\frac{1}{n^2}$

Setting $\frac{e^{-\frac{3}{2}}}{3}\cdot\frac{1}{n^2} < \frac{1}{2}\cdot 10^{-7}$,

$n^2 > \frac{2}{3}10^7e^{-\frac{3}{2}}$

$\Rightarrow n > 1219.645$

$\therefore$ At least 1220 subintervals is needed.