305

HW 7.1

5. $f(x+h)+f(x-h) = 2f(x) + f''(x)h^2 + \sum_{n=1}^{\infty} \frac{2f^{(2n+2)}}{(2n+2)!} h^{2n+2}$

$$\therefore\ f''(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2} + \sum_{n=1}^{\infty} \underbrace{\frac{2f^{(2n+2)}}{(2n+2)!}}_{a_{2n}} h^{2n}$$

$$\therefore\ a_{2n} = \frac{2f^{(2n+2)}(x)}{(2n+2)!}$$

Error term in (9) was derived in class.

6.

$$-f(x+2h) = -\cancel{f(x)} - f'(x)2h - \cancel{\tfrac{f''(x)}{2}4h^2} - \cancel{\tfrac{f^{(3)}(x)}{6}8h^3} - \cancel{\tfrac{f^{(4)}(x)}{24}16h^4} - \frac{f^{(5)}(\xi_1)}{120}32h^5$$

$$8f(x+h) = \cancel{8f(x)} + 8f'(x)h + \cancel{4f''(x)h^2} + \cancel{8\tfrac{f^{(3)}(x)}{6}h^3} + \cancel{8\tfrac{f^{(4)}(x)}{24}h^4} + \frac{8f^{(5)}(\xi_2)}{120}h^5$$

$$-8f(x-h) = -\cancel{8f(x)} + 8f'(x)h - \cancel{4f''(x)h^2} + \cancel{8\tfrac{f^{(3)}(x)}{6}h^3} - \cancel{8\tfrac{f^{(4)}(x)}{24}h^4} + \frac{8f^{(5)}(\xi_3)}{120}h^5$$

$$+)\ f(x-2h) = \cancel{f(x)} - f'(x)2h + \cancel{\tfrac{f''(x)}{2}4h^2} - \cancel{\tfrac{f^{(3)}(x)}{6}8h^3} + \cancel{\tfrac{f^{(4)}(x)}{24}16h^4} - \frac{f^{(5)}(\xi_4)}{120}32h^5$$

$$-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h) = 12f'(x)h$$

$$= 12f'(x)h + \frac{1}{15}\left[-4f^{(5)}(\xi_1)+f^{(5)}(\xi_2)+f^{(5)}(\xi_3)-4f^{(5)}(\xi_4)\right]h^5$$

(where $\xi_i \in](x-2h, x+2h)$, $i = 1,2,3,4$)

$$\therefore\ f'(x) = \frac{1}{12h}\left[-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)\right]$$

$$+ \underbrace{\frac{1}{180}\left[-4f^{(5)}(\xi_1)+f^{(5)}(\xi_2)+f^{(5)}(\xi_3)-4f^{(5)}(\xi_4)\right]h^4}_{\equiv\ \text{Error}}$$

$$\therefore\ |\text{Error}| \le \frac{1}{180}\left[4|f^{(5)}(\xi_1)| + |f^{(5)}(\xi_2)| + |f^{(5)}(\xi_3)| + 4|f^{(5)}(\xi_4)|\right]h^4$$

$$\le \frac{1}{18}\|f^{(5)}\|_\infty h^4$$

12. $L = \varphi(h) + a_1 h + a_3 h^3 + a_5 h^5 + \cdots$ — ①

$L = \varphi(\tfrac{h}{2}) + a_1\tfrac{h}{2} + a_3\tfrac{h^3}{8} + a_5\tfrac{h^5}{32} + \cdots$

$2L = \varphi(\tfrac{h}{2}) + a_1 h + a_3\tfrac{h^3}{4} + a_5\tfrac{h^5}{16} + \cdots$ — ②

② − ① : $L = \underbrace{\varphi(\tfrac{h}{2}) - \varphi(h)}_{\psi(h)} \underbrace{-\tfrac{3}{4}a_3}_{b_3}h^3 \underbrace{-\tfrac{15}{16}a_5}_{b_5}h^5 - \cdots$ — ③

$L = \psi(\tfrac{h}{2}) + \tfrac{b_3}{8}h^3 + \tfrac{b_5}{32}h^5 + \cdots$

$8L = 8\psi(\tfrac{h}{2}) + b_3h^3 + \tfrac{b_5}{4}h^5 + \cdots$ — ④

④ − ③ : $7L = 8\psi(\tfrac{h}{2}) - \psi(h) - \tfrac{3}{4}h^5 + \cdots$

$\therefore\ L = \tfrac{8}{7}\psi(\tfrac{h}{2}) - \tfrac{1}{7}\psi(h) - \tfrac{3}{28}h^5 + \cdots$ keep going…

13. $L = f(h) + C_6h^6 + C_9h^9 + \cdots$ — ⑤

$L = f(\tfrac{h}{2}) + C_6\tfrac{h^6}{64} + \tfrac{C_9}{512}h^9 + \cdots$

$64L = 64f(\tfrac{h}{2}) + C_6h^6 + \tfrac{C_9}{8}h^9 + \cdots$ — ⑥

⑥ − ⑤ : $63L = 64f(\tfrac{h}{2}) - f(h) - \tfrac{7}{8}C_9h^9 + \cdots$

$$\therefore\ L = \underbrace{\frac{64}{63}f(\tfrac{h}{2}) - \frac{1}{63}f(h)}_{\text{best estimate of } L} - \frac{1}{72}C_9h^9 + \cdots$$

14. $4f(x+h) = 4f(x) + 4f'(x)h + \cancel{2f''(x)h^2} + 4\frac{f^{(3)}(\xi_1)}{6}h^3$

$+)\ -f(x+2h) = -f(x) - 2f'(x)h - \cancel{2f''(x)h^2} - \frac{f^{(3)}(\xi_2)}{6}8h^3$

$$4f(x+h) - f(x+2h) = 3f(x) + 2hf'(x) + \frac{1}{3}\left[2f^{(3)}(\xi_1) - 4f^{(3)}(\xi_2)\right]h^3$$

$$\therefore\ f'(x) = \frac{1}{2h}\left[-3f(x)+4f(x+h)-f(x+2h)\right]$$

$$+ \underbrace{\frac{1}{6}\left[2f^{(3)}(\xi_1) - 4f^{(3)}(\xi_2)\right]h^2}_{\text{Error}}$$

$$|\text{Error}| \le \frac{1}{6}\left[2\|f^{(3)}\|_\infty + 4\|f^{(3)}\|_\infty\right]h^2 = \|f^{(3)}\|_\infty h^2$$

15. $f'(x) = \varphi(h) + a_2h^2 + a_4h^4 + \cdots$

where $\varphi(h) = \dfrac{f(x+h) - f(x-h)}{2h}$.

From (17) on P472,

$f'(x) = \psi(h) + O(h^4)$, where

$$\psi(h) = \frac{4}{3}\varphi(\tfrac{h}{2}) - \frac{1}{3}\varphi(h)$$

$$= \frac{4}{3}\,\frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h} - \frac{1}{3}\,\frac{f(x+h)-f(x-h)}{2h}$$

$$= \frac{8f(x+\frac{h}{2}) - 8f(x-\frac{h}{2}) - f(x+h) + f(x-h)}{6h}$$

$$= \frac{f(x-h) - 8f(x-\frac{h}{2}) + 8f(x+\frac{h}{2}) - f(x+h)}{6h}$$