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HW 6.4

5. Yes

$\left(\because \; f, f' \in C(-\infty, \infty) \right.$
$\left. \& \; f \text{ is piecewise } \mathbb{P}_2 \right)$

6. No

$$\left(\because f''(x) = \begin{cases} 0 & (-\infty, 1) \\ -1 & (1,2) \\ 0 & (2, \infty) \end{cases} \rightarrow \text{not conti on } (-\infty, \infty) \right)$$

7. From the conti of f, f', & f'' we get $a = c = d$

$$\therefore f(x) = \begin{cases} a(x-2)^2 + b(x-1)^3, & x \in (-\infty, 1] \\ a(x-2)^2 & x \in [1,3] \\ a(x-2)^2 + e(x-3)^3, & x \in [3, \infty) \end{cases}$$

$f(1) = 7 \Rightarrow \underline{a = 7}$

$f(0) = 26 \Rightarrow 7 \cdot 4 - b = 26 \Rightarrow \underline{b = 2}$

$f(4) = 25 \Rightarrow 7 \cdot 4 + e = 25 \Rightarrow \underline{e = -3}$

$\therefore a = 7, \; b = 2, \; c = 7, \; d = 7, \; e = -3$

$$\text{or } f(x) = \begin{cases} 7(x-2)^2 + 2(x-1)^3, & x \in (-\infty, 1] \\ 7(x-2)^2 & x \in [1,3] \\ 7(x-2)^2 - 3(x-3)^3, & x \in [3, \infty) \end{cases}$$

12. $$f(x) = \begin{cases} x^3 + x, & x \le 0 \\ x^3 - x, & x \ge 0 \end{cases} \Rightarrow f \text{ is conti on } (-\infty, \infty)$$

$$f'(x) = \begin{cases} 3x^2 + 1, & x > 0 \\ 3x^2 - 1, & x < 0 \end{cases} \Rightarrow f' \text{ is not cont at } x = 0$$

$$f''(x) = \begin{cases} 6x & x > 0 \\ 6x & x < 0 \end{cases}$$

$$\therefore \lim_{x \to 0^+} f''(x) = \lim_{x \to 0^-} f''(x) = 0$$

But f is <u>not</u> a cubic spline

14. $$f(x) = \begin{cases} 2(x+1) + (x+1)^3 & [-1, 0] \\ 3 + 5x + 3x^2 & [0, 1] \\ 11 + 11(x-1) + 3(x-1)^2 - (x-1)^3 & [1, 2] \end{cases} \Rightarrow \text{conti on } [-1, 2]$$

$$f'(x) = \begin{cases} 2 + 3(x+1)^2 & [-1, 0] \\ 5 + 6x & [0, 1] \\ 11 + 6(x-1) - 3(x-1)^2 & [1, 2] \end{cases} \Rightarrow \text{conti on } [-1, 2]$$

$$f''(x) = \begin{cases} 6(x+1) & [-1, 0] \\ 6 & [0, 1] \\ 6 - 6(x-1) & [1, 2] \end{cases} \Rightarrow \text{conti on } [-1, 2]$$

$f''(-1) = f''(2) = 0$

$\therefore$ f is a natural cubic spline

19. $$S(x) \equiv \begin{cases} S_0(x), & -1 \le x \le 0 \\ S_1(x), & 0 \le x \le 1 \end{cases} \qquad S_i \in \mathbb{P}_3, \; i = 0, 1$$

Let $a \equiv \lim_{x \to 0^+} S_0''(x) = \lim_{x \to 0^-} S_1''(x)$

$S_0''(-1) = 0, \; S_0''(0) = a \Rightarrow S_0''(x) = ax + a = a(x+1)$

$\therefore S_0(x) = \frac{a}{6}(x+1)^3 + b(x+1) + c(-x)$

$\underline{5 = S_0(-1) = c} \qquad 7 = S_0(0) = \frac{a}{6} + b \Rightarrow \underline{a + 6b = 42}$ — ①

$S_1''(a) = a, \; S_1''(1) = 0 \Rightarrow S_0''(x) = -ax + a = a(1-x)$

$\therefore S_1(x) = \frac{a}{6}(1-x)^3 + dx + e(1-x)$

$\underline{9 = S_1(1) = d}, \quad 7 = S_1(0) = \frac{a}{6} + e \Rightarrow \underline{a + 6e = 42}$ — ②

① − ② $\Rightarrow 6b - 6e = 0 \Rightarrow \underline{b = e}$

$\therefore S_0(x) = \frac{a}{6}(x+1)^3 + b(x+1) - 5x$

$S_0'(x) = \frac{a}{2}(x+1)^2 + b - 5$

$S_1(x) = \frac{a}{6}(1-x)^3 + 9x + b(1-x)$

$S_1'(x) = -\frac{a}{2}(1-x) + 9 - b$

$S_0'(0) = \frac{a}{2} + b - 5 = S_1'(0) = -\frac{a}{2} + 9 - b$

$\Rightarrow \underline{a + 2b = 14}$ — ③

① − ③ $\Rightarrow 4b = 28 \quad \therefore \underline{b = 7} \Rightarrow \underline{a = 0}$

$$S(x) = \begin{cases} S_0(x) = 7(x+1) - 5x = 2x + 7, & -1 \le x \le 0 \\ S_1(x) = 9x + 7(1-x) = 2x + 7 & 0 \le x \le 1 \end{cases}$$

$\therefore S(x) = 2x + 7$ on $[-1, 1]$!!

(Rmk: Given data $(-1,5), (0,7), (1,9)$ are colinear. That is why $S(x) \in \mathbb{P}_1[-1,1]$)