305 HW 6.3

1. $P_4(x) = 2 - 9x + 3x^2 + 7x^2(x-1) + 5x^2(x-1)^2$

2. $P_5(x) = 2 - 9x + 3x^2 + 7x^2(x-1) + 5x^2(x-1)^2 + C\,x^2(x-1)^2(x-2)$

$2 = P(3) = 2 - \cancel{27} + \cancel{27} + 7\cdot 9\cdot 2 + 5\cdot 9\cdot 4 + C\cdot 9\cdot 4\cdot 1$

$\cancel{2} = \cancel{2} + 126 + 180 + 36C$

$\therefore 36C = -306 \Rightarrow C = -\frac{306}{36} = -\frac{17}{2}$

$\therefore C$

$\therefore P_5(x) = 2 - 9x + 3x^2 + 7x^2(x-1) + 5x^2(x-1)^2 - \frac{17}{2}x^2(x-1)^2(x-2)$

7. Let $P \in \mathbb{P}_n$ interpolate f at $x_0, \underset{=x_1}{x_0}, \underset{=x_2}{x_0}, \cdots, \underset{=x_{k-1}}{x_0}$ (k repetition). Then,

$$P(x) = \sum_{j=0}^{k-1} \underbrace{f[x_0, x_0, \cdots, x_0]}_{j+1 \text{ repetition}} \underbrace{\prod_{i=0}^{j-1}(x - x_i)}_{=(x-x_0)^j}$$

(7) on P343 $\Rightarrow f[x_0, x_0, \cdots, x_0] = \frac{1}{j!} f^{(j)}(x_0)$

$$\therefore P(x) = \sum_{j=0}^{k-1} \frac{1}{j!} f^{(j)}(x_0)(x - x_0)^j$$

8. Suppose that $x_0, x_1, \cdots, x_n$ has $m+1$ distinct pts $y_0, y_1, y_2 \cdots, y_m$ ($\therefore m \le n$) and y_k is repeated l_k times

$\therefore l_k \ge 1 \quad \forall\, 0 \le k \le m \quad \& \quad \sum_{k=0}^{m} l_k = n+1$

Now, we want to prove that for a polynomial p

p interpolates 0 at $x_0, x_1, \cdots, x_n$

$\Leftrightarrow \prod_{j=0}^{n}(x - x_j)$ is a factor of P.

$(\Leftarrow)$ $\prod_{j=0}^{n}(x - x_j) = \prod_{k=0}^{m}(x - y_k)^{l_k}$ devides P

$\Rightarrow P(x) = g(x)\prod_{k=0}^{m}(x - y_k)^{l_k}$ for some polynomial g.

$\therefore P^{(l_k - 1)}(y_k) = 0 \quad \forall\, k = 0, 1, 2, \cdots, m$

$\therefore$ P interpolates 0 at $x_0, x_1, \cdots, x_n$

$(\Rightarrow)$ Let k be any given integer s.t. $0 \le k \le m$

By Taylor expansion of f at y_k,

$$P(x) = \sum_{i=0}^{M} \frac{f^{(i)}(y_k)}{i!}(x - y_k)^i$$

Since P interpolates $x_0, x_1, \cdots, x_n$ & y_k occurs l_k times, $f^{(i)}(y_k) = 0, \ \forall\, 0 \le i \le l_k - 1$

$\therefore P(x) = \sum_{i=l_k}^{M} \frac{f^{(i)}(y_k)}{i!}(x - y_k)^i = (x - y_k)^{l_k}\cdot(\text{polynomial})$

$\therefore (x - y_k)^{l_k}$ is a factor of P, $\forall\, k = 0, 1, \cdots, m$

$\therefore \prod_{k=0}^{m}(x - y_k)^{l_k} = \prod_{j=0}^{n}(x - x_j)$ is a factor of P ///

9. $(f - g)$ interpolates 0 at $x_0 \cdots x_n$

h ________________

$\Rightarrow \prod_{i=0}^{n}(x - x_i)$ is a factor of $f - g$ & h

$\therefore \prod_{i=0}^{n}(x - x_i)$ ________ of $f - g + ch$ $\forall$ constant c

$\therefore (f + ch) - g$ interpolates 0 at $x_0, x_1, \cdots, x_n$

$\Leftrightarrow f + ch$ interpolates 0 at $x_0, x_1, \cdots, x_n$

11. Trivial by the original definition

15. Skip.