305

HW 6.1

1. (a) $P_1(x) = 5 - \frac{3}{2}(x-3)$

(b) $P_2(x) = 146 + 24(x-7) + 5(x-7)(x-1)$

(c) Using the result of 1(b)

$P_3(x) = P_2(x) + c(x-7)(x-1)(x-2)$

$x=3 \rightarrow \quad 10 = 146 - 96 - 40 - 8c$

$\Rightarrow c = 0$

$\therefore P_3(x) = 146 + 24(x-7) + 5(x-7)(x-1)$

(or $P_3(x) = 10 + 34(x-3) + 5(x-3)(x-7)$)

2. For given fcns f & g and constants a & b

$$L(af+bg) = \sum_{i=0}^{n} (af+bg)(x_i)\, l_i(x)$$

$$= \sum_{i=0}^{n} \left(af(x_i) + bg(x_i)\right) l_i(x)$$

$$= a\sum_{i=0}^{n} f(x_i) l_i(x) + b\sum_{i=0}^{n} g(x_i) l_i(x)$$

$$= aL(f) + bL(g)$$

$\therefore L$ is linear

4. (i) $Lq,\ q \in \mathbb{P}_n$

(ii) Both Lq & q interpolates q at $x_0, x_1, \cdots, x_n$

By (i), (ii), & the uniqueness of the interpolation in $\mathbb{P}_n$,

$$Lq = q$$

for all $q \in \mathbb{P}_n$

7. $k = 1, 2, 3, \cdots, n$

$\begin{pmatrix}\text{\# of long}\\ \text{operations}\end{pmatrix} = 1, 3, 5, \cdots, 2n-1$

$\therefore$ Total # of long operations

$= 1 + 3 + 5 + \cdots + 2n-1$

$= \sum_{i=1}^{n}(2i-1) = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$

$= \cancel{2}\frac{n(n+1)}{\cancel{2}} - n$

$= n^2$

10. $$P(x) = \sum_{i=0}^{n} y_i\, l_i(x)$$

$$= \sum_{i=0}^{n} y_i \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

$$= \sum_{i=0}^{n} \left(y_i \prod_{\substack{j=0\\ j\neq i}}^{n} (x_i-x_j)^{-1} \right) \underbrace{\prod_{\substack{j=0\\ j\neq i}}^{n} (x-x_j)}_{x^n + (\ )x^{n-1} + \cdots}$$

$\therefore$ The coefficient of x^n in $P(x)$ is

$$\sum_{i=0}^{n} y_i \prod_{\substack{j=0\\ j\neq i}}^{n} (x_i - x_j)^{-1}$$

11. Let $q \in \mathbb{P}_{n-1}$. Then

$$Lq = \sum_{i=0}^{n} q(x_i)\, l_i(x) \equiv q(x) \quad \text{— } (*)$$

(from #4)

From #10 & $(*)$ the coefficient of x^n of $q(x)$ is

$$\sum_{i=0}^{n} q(x_i) \prod_{\substack{j=0\\ j\neq i}}^{n} (x_i - x_j)^{-1}$$

Since $q \in \mathbb{P}_{n-1}$, it should be 0.

$$\therefore \sum_{i=0}^{n} q(x_i) \prod_{\substack{j=0\\ j\neq i}}^{n} (x_i - x_j)^{-1} = 0$$

13. $$|f(x) - P(x)| = \frac{1}{23!} \left| f^{(23)}(\xi_x) \prod_{i=0}^{n} (x - x_i) \right|$$

$$|\cosh x - P(x)| \le \frac{|\sinh \xi_x|}{23!}\, 2^{23} \qquad \begin{pmatrix} \because x_i \in [-1,1], \forall i \\ \Rightarrow |x - x_i| \le 2, \forall i \end{pmatrix}$$

$$\frac{|\cosh x - P(x)|}{\cosh x} \le \frac{2^{23}}{23!} \left| \frac{\sinh \xi_x}{\cosh x} \right|$$

$$\le \frac{2^{23} \cdot \sinh 1}{23!} \approx 3.81 \times 10^{-16} < 5 \times 10^{-16}$$

22. (Lagrange form)

$$P(x) = 0 + l_1(x) - l_2(x)$$

$$= \frac{(x+2)(x-1)}{-2} - \frac{x(x+2)}{3}$$

(Newton form)

$$P(x) = \frac{1}{2}(x+2) - \frac{5}{6}(x+2)x$$

26.

x	0	$\frac{1}{2}$	1
y	-1	$\frac{1}{6}$	$\frac{8}{9}$

$$P_2(x) = -1 + \frac{7}{3}x - \frac{8}{9}x\left(x - \frac{1}{2}\right)$$

$$= -1 + \frac{7}{3}x - \frac{8}{9}x^2 + \frac{4}{9}x$$

$$= -\frac{8}{9}x^2 + \frac{25}{9}x - 1$$

$P_2(x) = 0 \iff 8x^2 - 25x + 9 = 0$

$$\therefore x = \frac{25 \pm \sqrt{25^2 - 4\cdot 8\cdot 9}}{16} = \frac{25 \pm \sqrt{337}}{16}$$

$$\approx 0.4151525,\ 2.70985$$

$\therefore x \approx 0.4151525$

30. Assume that we have $n+1$ distinct nodes $x_0, x_1, x_2, \cdots, x_n$

g interpolates f at $x_0, x_1, \cdots, x_{n-1}$
$\Rightarrow g(x_i) = f(x_i) \quad (0 \le i \le n-1)$ — (1)

$h(x_i) = 0, \quad (0 \le i \le n-1)$
$h(x_n) = 1$ — (2)

Choose $c = f(x_n) - g(x_n)$. Then

$$[g + ch](x_i) = g(x_i) + c\cdot h(x_i) \overset{(1),(2)}{=} f(x_i) \quad \forall\, 0 \le i \le n-1 \quad \text{— } (3)$$

$$[g + ch](x_n) = g(x_n) + c\, \cancel{h(x_n)}^{1}$$

$$= g(x_n) + \left(f(x_n) - g(x_n)\right)$$

$$= f(x_n) \quad \text{— } (4)$$

By (3) & (4),

$g + (f(x_n) - g(x_n))\cdot h$ interpolates f at $x_0, x_1, x_2, \cdots, x_n$