HW 3.2

7. Note that

$$\begin{cases} x_{n+1} = x_n - \dfrac{\tan x_n - 1}{\sec^2 x_n}, & n \geq 0 \\ x_0 = 0 \end{cases}$$

is the seq generated by Newton's method applied to the eq

$$\underbrace{\tan x - 1}_{\equiv f(x)} = 0$$

with the initial guess $x_0 = 0$

Since $f(0) = -1 < 0 < f(1)$,

& f is $\uparrow$ on $\mathbb{R}$,

$\exists!$ root of the eq $f(x) = 0$ in $(0, 1)$, which is $\frac{\pi}{4}$

Note that $x_1 = 1$ & f is convex and $\uparrow$ on $(0, \frac{\pi}{2})$, by a modified version of Thm 2 $\{x_n\}$ conv to the root, i.e.,

$$\lim_{n\to\infty} x_n = \frac{\pi}{4}$$

8. $f(x) = 4x^3 - 2x^2 + 3$

$f'(x) = 12x^2 - 4x$

$$\begin{cases} x_{n+1} = x_n - \dfrac{4x_n^3 - 2x_n^2 + 3}{12x_n^2 - 4x_n} \\ x_0 = -1 \end{cases}$$

$x_1 = -0.8125$

$x_2 = -0.770804$

$x_3 = -0.768832$

$x_4 = -0.768828$

9. $f(x) = x^3 - 2$, $f'(x) = 3x^2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)}$$

$$= 1 - \frac{-1}{3} = \frac{4}{3}$$

$$x_2 = x_1 - \frac{f(\frac{4}{3})}{f'(\frac{4}{3})} = \frac{4}{3} - \frac{\frac{64}{27} - 2}{\frac{16}{3}}$$

$$= \frac{4}{3} - \frac{5}{72} = \frac{91}{72} \approx 1.26389$$

10. $x = \sqrt[3]{R}$ if $x^3 = R$

$\therefore$ $f(x) = 0$ where $f(x) = x^3 - R$

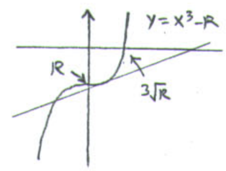
$f'(x) = 3x^2$

$$\therefore x_{n+1} = x_n - \frac{x_n^3 - R}{3x_n^2} = x_n - \frac{1}{3}x_n + \frac{R}{3x_n^2}$$

$$= \frac{2}{3}x_n + \frac{R}{3x_n}$$

The values of x_0 for which $\lim_{n\to\infty} x_n = \sqrt[3]{R}$

(Note that $R > 0$)



If $0 < x_0 < \sqrt[3]{R}$ then

$$\begin{cases} x_1 > \sqrt[3]{R}, \\ x_{n+1} < x_n, \ \forall n \geq 1, \\ x_n \to \sqrt[3]{R}, \text{ as } n \to \infty \end{cases}$$

If $x_0 > \sqrt[3]{R}$ then

$$\begin{cases} x_{n+1} < x_n, \ \forall n \geq 0 \\ x_n \to \sqrt[3]{R}, \text{ as } n \to \infty \end{cases}$$

$\therefore$ Any $x_0 > 0$ gives the convergence

However, a slightly greater # than $\sqrt[3]{R}$ is an ideal choice for x_0.

11. $x = \sqrt[5]{R}$, $R > 0$

$\Rightarrow x^5 = R \iff x^5 - R = 0$

Set $f(x) = x^5 - R$

$\therefore f'(x) = 5x^4$

$$\therefore x_{n+1} = x_n - \frac{x_n^5 - R}{5x_n^4}$$

$$= \frac{4}{5}x_n + \frac{R}{5x_n^4} \quad \forall n \geq 0$$

Choose a real # greater than $\sqrt[5]{R}$, then $\lim_{n\to\infty} x_n = \sqrt[5]{R}$

($\because$ same argument as in #10)

14. $f(r) = f'(r) = 0 \neq f''(r)$

& $f \in C^2$

$e_n = x_n - r$

By (2) on P84

$$e_{n+1} = \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)}$$

$$= \frac{\frac{1}{2} f''(\xi_n) e_n^2}{f'(x_n)} \quad \text{— ①}$$

where $\xi_n \in I(x_n; x_n - e_n) = I(x_n, r)$

Also, by applying Taylor's Thm on $f'(x)$,

$0 = f'(r) = f'(x_n - e_n) = f'(x_n) - e_n f''(\zeta_n)$

where $\zeta_n \in I(x_n, r)$

$\therefore$ $f'(x_n) = e_n f''(\zeta_n)$ — ②

By ① & ②, we get

$$e_{n+1} = \frac{\frac{1}{2} f''(\xi_n) e_n^2}{f''(\zeta_n) e_n}$$

$$= \frac{1}{2} \frac{f''(\xi_n)}{f''(\zeta_n)} e_n$$

$$\approx \frac{1}{2} e_n,$$

if x_n is close enough to r

$\therefore$ $x_n \to r$ linearly