HW 2.2

1. $y = x - \sin x$

$$= \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots + \underset{\substack{\uparrow \\ 7\text{th}}}{\frac{x^{15}}{15!}} - \underset{\substack{\uparrow \\ 8\text{th}}}{\frac{x^{17}}{17!}} + \cdots$$

$$|E_{14}(x)| = \left|\frac{\cos\xi}{15!}x^{15}\right|, \quad |E_{16}(x)| = \left|\frac{\cos\xi}{17!}x^{17}\right|$$

$$\therefore |E_{14}(1.9)| \le \frac{(1.9)^{15}}{15!} \approx 1.16\times 10^{-8}$$

$$|E_{16}(1.9)| \le \frac{(1.9)^{17}}{17!} \approx 1.54\times 10^{-10}$$

$\therefore$ We need at least seven terms to have error less than 10^{-9}

2. $y \equiv \sin x < x, \ \forall x > 0$

$$1 - \frac{y}{x} = 1 - \frac{\sin x}{x}$$

If $x = \frac{1}{2}$

$$1 - \frac{\sin(\frac{1}{2})}{\frac{1}{2}} \approx 0.04115$$

$$2^{-5} = 0.03125, \quad 2^{-4} = 0.0625$$

$$\therefore 2^{-(4+1)} < 1 - \frac{\sin(\frac{1}{2})}{\frac{1}{2}} < 2^{-4}$$

$\therefore$ 4 or 5 significant binary digits are lost.

5. (i) $1 - \cos x$

$$= 2\cdot\frac{1-\cos(2\cdot\frac{x}{2})}{2}$$

$$= 2\cdot\sin^2\left(\frac{x}{2}\right)$$

(ii) $1 - \cos x$

$$= \frac{(1-\cos x)(1+\cos x)}{1+\cos x}$$

$$= \frac{1-\cos^2 x}{1+\cos x} = \frac{\sin^2 x}{1+\cos x}$$

6. $\sqrt{x^4+4} - 2$

$$= \frac{(\sqrt{x^4+4}-2)(\sqrt{x^4+4}+2)}{\sqrt{x^4+4}+2}$$

$$= \frac{(x^4+4) - 4}{\sqrt{x^4+4}+2}$$

$$= \frac{x^4}{\sqrt{x^4+4}+2}$$

7. When $x \approx 0$,

$$e^x \approx 1 \approx e^{-x}$$

$\therefore\ \exists$ cancellation error in

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

Note that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$e^x - e^{-x} = 2x + 2\cdot\frac{x^3}{3!} + 2\cdot\frac{x^5}{5!} + \cdots$$

$$= 2\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]$$

$$\therefore \sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$\therefore$ Use the following approximation

$$\sinh x \approx \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}$$

to evaluate $\sinh x$ for $|x| \ll 1$ with appropriate n depending on the required accuracy

8. Let α & β be two roots of

$$ax^2 + bx + c = 0$$

i.e., $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ — ①

$\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ — ②

If $b > 0$, $\exists$ cancellation error in ①

If $b < 0$, $\exists$ ________ in ②

Note that $\alpha\beta = \dfrac{c}{a}$

If $b>0$, calculate β first using ② then calculate α using

$$\alpha = \frac{c}{a\beta}$$

If $b<0$, calculate α first using ① then calculate β using

$$\beta = \frac{c}{a\alpha}$$

9. (a) $\sqrt{x^2+1} - x = \dfrac{1}{\sqrt{x^2+1}+x}$

(when $x > 0$ & $|x| \ll 1$)

(b) When $x \approx y > 0$

$$\log x - \log y = \log\frac{x}{y}$$

(c) When $|x| \ll 1$

$$x^{-3}(\sin x - x)$$

$$= x^{-3}\left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$\approx -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \frac{x^6}{9!} + \cdots + \frac{(-1)^{n+1}x^{2n}}{(2n+3)!}$$