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HW 2.1

1. $2 \cdot \varepsilon = 2 \cdot 2^{-24} = 2^{-23}$

2. $\frac{1}{10} = (0.a_1a_2a_3 \cdots)_2 \equiv x$

$\frac{2}{10} = (a_1.a_2a_3 \cdots)_2 \Rightarrow a_1 = 0$

$\frac{4}{10} = (a_2.a_3a_4 \cdots)_2 \Rightarrow a_2 = 0$

$\frac{8}{10} = (a_3.a_4a_5 \cdots)_2 \Rightarrow a_3 = 0$

$\frac{16}{10} = (a_4.a_5a_6 \cdots)_2 \Rightarrow a_4 = 1$

$\frac{6}{10} = (0.a_5a_6 \cdots)_2$

$\frac{12}{10} = (a_5.a_6a_7 \cdots)_2 \Rightarrow a_5 = 1$

$\frac{2}{10} = (0.a_6a_7 \cdots)_2$

$= (0.a_2a_3a_4 \cdots)_2$

$\therefore a_6 = a_2 = 0, \quad a_7 = a_3 = 0$

$a_8 = a_4 = 1, \quad a_9 = a_5 = 1$

$\cdots$

$\therefore \frac{1}{10} = (0.0\overline{0011}\,\overline{0011})_2$

$= (1.\overline{1001}\overline{100})_2 \times 2^{-24}$

$= (1.1001 \cdots 100\underset{20}{1}10\underset{23}{0}1\overline{1001})_2 \times 2^{-4}$

$fl(x) = x_+ = (1.1001 \cdots 10011\underset{23}{0}1)_2 \times 2^{-4}$

$x_- = (1.1001 \cdots 100110\underset{23}{0})_2 \times 2^{-4}$

$x - x_- = (0.00 \cdots \underset{23}{0}\,\overline{1100})_2 \times 2^{-4}$

$= (1.\overline{10011})_2 \times 2^{-4} \times 2^{-24}$

$= \frac{1}{10} \times 2^{-24}$

$x_+ - x = (x_+ - x_-) - (x - x_-) = 2^{-4} \times 2^{-23}$

$= 2^{-4} \times 2^{-23} - \frac{1}{10} \cdot 2^{-24}$

$= (\frac{1}{8} - \frac{1}{10}) \times 2^{-24} = \frac{2}{80} \times 2^{-24}$

$= \frac{1}{10} \times 2^{-26}$

$\therefore fl(x) = x_+$

round off error $= x_+ - x = \frac{1}{10} \times 2^{-26}$

rel. roundoff error $= \left|\frac{x_+ - x}{x}\right| = 2^{-26}$

7. Any positive machine # in Marc-32

$: \underbrace{(1.a_1a_2a_3 \cdots a_{23})_2}_{2^{23}\text{ choices}} \times 2^m$

$-126 \le m \le 127 \quad : 254 = 2^8 - 2 = 2(2^7 - 1)$

$\therefore \quad 2^{23} \cdot 2(2^7 - 1) = 254 \times 2^{23}$

For all nonzero machine #'s

$2 \cdot 2^{24}(2^7 - 1) = 2^{25}(2^7 - 1)$

$= 2^{32} - 2^{25} = 254 \times 2^{24}$

$= 4{,}261{,}412{,}864 \quad (\approx 2^{32})$

10. $x = (1.00 \cdots 001\underset{22\text{th}}{0}|0\underset{25\text{th}}{1})_2 \times 2^3$

$= 2^3 + 2^{-19} + 2^{-22}$

$x_- = (1.00 \cdots 001\underset{22\text{th}}{0})_2 \times 2^3$

$= 2^3 + 2^{-19}$

$x_+ = (1.00 \cdots 0011)_2 \times 2^3$

$= 2^3 + 2^{-19} + 2^{-20}$

$\left.\begin{array}{l} x - x_- = 2^{-22} \\ x_+ - x = 2^{-20} - 2^{-22} \end{array}\right\} \quad x - x_- < x_+ - x$

$\therefore fl(x) = x_- = 2^{-22}$

$|x - fl(x)| = x - x_- = 2^{-22}$

$\frac{|x - fl(x)|}{|x|} = \frac{2^{-22}}{2^3 + 2^{-19} + 2^{-22}} < \frac{2^{-22}}{2^3} = 2^{-25} \le 2^{-24}$

12. $x = (1.11 \cdots \underset{23}{1}1\underset{25}{1}1)_2 \times 2^{-1}$

$x^* = (10.00 \cdots 00\overset{\downarrow}{0})_2 \times 2^{-1}$

$= (1.00 \cdots 0)_2 \times 2^0 = 1$

$\therefore x^* - x = 1 - \sum_{n=1}^{26} 2^{-n} = 1 - \frac{\cancel{\frac{1}{2}}(1 - (\frac{1}{2})^{26})}{\cancel{1 - \frac{1}{2}}}$

$= 1 - 1 + (\frac{1}{2})^{26}$

$= 2^{-26}$

14. (a), (b), (d)

(c) $\frac{xy}{1+\delta} = xy(1 - \delta + \delta^2 - \delta^3 + \cdots)$

$= xy(1 + \tilde{\delta} + O(\varepsilon^2))$

$\therefore \frac{xy}{1+\delta} = xy(1 + \tilde{\delta}), \quad |\tilde{\delta}| \le \varepsilon = 2^{-24}$

$\therefore$ true

(e) $fl(x+y+z) = (x+y+z)(1+\delta_1)(1+\delta_2)$

$= (x+y+z)(1+2\delta)$

$|\delta| \le \varepsilon = 2^{-24}$

$\therefore$ False

16. (a) No (b) No (c) No (d) No

(e) Yes $\left(\frac{1}{256} = 2^{-8} = (1.00 \cdots 0)_2 \times 2^{-8}\right)$

26. $2^n = (1.00 \cdots 0)_2 \times 2^n$

$2^{n+1} = (1.00 \cdots 0)_2 \times 2^{n+1}$

$= (10.00 \cdots 0)_2 \times 2^n$

$2^n < (1.a_1a_2 \cdots a_3)_2 \times 2^n < 2^{n+1}$

$\forall a_i$'s except $a_1 = a_2 = \cdots = a_3 = 0$

$\therefore$ $2^{23} - 1$ choices

28. (i) regular normalization with $n = 48$

$\varepsilon = \frac{1}{2} \cdot 2^{1-n} = 2^{-n} = 2^{-48}$

(ii) left shifted normalization with $n = 48$

$\varepsilon = \frac{1}{2} \cdot 2^{-n} = 2^{-n-1} = 2^{-49}$

36. Discard this problem from the HW list.