HW 1.2

1. $f(x) = \tan^{-1} x \qquad f(0) = 0$

$f'(x) = \frac{1}{1+x^2} \qquad f'(0) = 1$

$f''(x) = \frac{-2x}{(1+x^2)^2} \qquad f''(0) = 0$

$f^{(3)}(x) = \frac{6x^2-2}{(1+x^2)^3} \qquad f^{(3)}(0) = -2$

$\therefore \tan^{-1} x = x + \frac{1}{6} \cdot \frac{6\xi^2 - 2}{(1+\xi^2)^3} \cdot x^3$

for some $\xi \in I(0, x)$

$\therefore \tan^{-1} x = x + O(x^k)$

for all $k \leq 3$

$\therefore$ Best integer value $= 3$

6. (a) F (b) F (c) F (d) T (e) F

7. (d)

14. $\frac{d}{dx}(2x^3y^2 + x^2y + e^x) = \frac{d}{dx}(C)$

$\Rightarrow 6x^2y^2 + 4x^3y\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} + e^x = 0$

$(4x^3y + x^2)\frac{dy}{dx} = -(6x^2y^2 - 2xy - e^x)$

$\therefore \frac{dy}{dx} = -\frac{6x^2y^2 - 2xy - e^x}{4x^3y + x^2}$

19. (a) $\frac{\pi}{2}$ (b) 1

(c) 0 (d) 1

20. $e^x \geq 0, \ \forall x$

$\therefore \int_0^{\frac{\pi}{2}} e^x \cos x \, dx = e^{\xi} \underbrace{\int_0^{\frac{\pi}{2}} \cos x \, dx}_{=1}$

for some $\xi \in (0, \frac{\pi}{2})$

Set $y \equiv \xi$ to get

$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx = e^y$

for some $y \in (0, \frac{\pi}{2})$

28. $x_n = x + o(1)$

$\Leftrightarrow x_n - x = o(1)$

$\Leftrightarrow \lim_{n\to\infty} \frac{|x_n - x|}{1} = 0$

$\Leftrightarrow \lim_{n\to\infty} x_n = x$

32. $x_n = O(\alpha_n)$

$\Leftrightarrow \exists K > 0, M > 0$ s.t.

$\forall n \geq K, \ |x_n| \leq M|\alpha_n|$

$\Rightarrow \forall n \geq K, \ |cx_n| \leq M|c||\alpha_n|$

$\therefore cx_n = O(\alpha_n)$

33. $x_n = O(\alpha_n)$

$\Leftrightarrow \exists K > 0, M > 0$ s.t.

$\forall n \geq K, \ |x_n| \leq M|\alpha_n|$

$\Rightarrow \forall n \geq K, \ \left|\frac{x_n}{\log n}\right| \leq \frac{M}{|\log n|}|\alpha_n|$

& $\frac{M}{|\log n|} \to 0$ as $n \to \infty$

$\therefore \frac{x_n}{\log n} = o(\alpha_n)$

34. $\cos x - 1 + \frac{x^2}{2} = \frac{\cos\xi}{4!}x^4$

for some $\xi \in (0, x)$.

$\therefore \left|\cos x - 1 + \frac{x^2}{2}\right| \leq \frac{1}{24}|x^4|$

$\therefore \cos x - 1 + \frac{x^2}{2} = O(x^\beta)$, as $x \to 0$

for all $\beta \leq 4$.

35. Suppose that $x_n = o(\alpha_n)$

i.e., $\lim_{n\to\infty} \left|\frac{x_n}{\alpha_n}\right| = 0$

$\therefore \forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t.

$\left|\frac{x_n}{\alpha_n}\right| \leq \varepsilon \quad \forall n \geq K$

$\therefore |x_n| \leq \varepsilon|\alpha_n|, \ \forall n \geq K$

i.e., $x_n = O(\alpha_n)$

However, the converse is not true

$\frac{1}{n+1} = O(\frac{1}{n})$

$(\because \left|\frac{1}{n+1}\right| \leq 1 \cdot \left|\frac{1}{n}\right|, \ \forall n \in \mathbb{N})$

but $\frac{1}{n+1} \neq o(\frac{1}{n})$

$\left(\because \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0\right)$