4. $\lim_{x\to 0}\frac{x-\sin x}{x^3} \overset{(H)}{=} \lim_{x\to 0}\frac{1-\cos x}{3x^2}$
$\overset{(H)}{=} \lim_{x\to 0}\frac{\sin x}{6x} = \lim_{x\to 0}\frac{\cos x}{6} = \frac{1}{6}$

Define $f(x) = \begin{cases} \frac{x-\sin x}{x^3}, & \text{if } x\neq 0 \\ \frac{1}{6}, & \text{if } x=0 \end{cases}$

Then $f$ is continuous

Obviously, $f$ is diff at $x\neq 0$.

$\frac{f(x)-f(0)}{x} = \frac{\frac{x-\sin x}{x^3}-\frac{1}{6}}{x} = \frac{6x-6\sin x - x^3}{6x^4}$.

$\lim_{x\to 0}\frac{f(x)-f(0)}{x} \overset{(H)}{=} \lim_{x\to 0}\frac{6-6\cos x-3x^2}{24x^3}$.

$\overset{(H)}{=} \lim_{x\to 0}\frac{6\sin x-6x}{72x^2} \overset{(H)}{=} \lim_{x\to 0}\frac{6\cos x-6}{144x}$

$\overset{(H)}{=} \lim_{x\to 0}\frac{-6\sin x}{144} = 0$

$\therefore f'(0)=0 \quad \therefore f\in C^1(\mathbb{R})$

9. $\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$
$= \lim_{h\to 0}\frac{f(x+h)-f(x)+f(x)-f(x-h)}{2h}$
$= \lim_{h\to 0}\left[\frac{1}{2}\frac{f(x+h)-f(x)}{h} + \frac{1}{2}\cdot\frac{f(x-h)-f(x)}{-h}\right]$
$= \frac{1}{2}f'(x)+\frac{1}{2}f'(x) = f'(x)$

13. $f(x) = x^2-2x+3$
$f(3)-f(1) = 6-2=4$
$f'(\xi) = 2\xi-2$
$\therefore f(3)-f(1) = f'(\xi)(3-1)$
$\Leftrightarrow 4 = (2\xi-2)2$
$1 = \xi-1$
$\therefore \underline{\xi=2}$

15. $f(x) = \cosh x = \frac{1}{2}(e^x+e^{-x})$

$$\begin{aligned} e^x &= 1+\cancel{x}+\frac{x^2}{2!}+\cancel{\frac{x^3}{3!}}+\frac{x^4}{4!}+\cdots \\ +)\ e^{-x} &= 1-\cancel{x}+\frac{x^2}{2!}-\cancel{\frac{x^3}{3!}}+\frac{x^4}{4!}-\cdots \\ \hline &= 2\left[1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right] \end{aligned}$$

$\therefore \cosh x = \frac{1}{2}(e^x+e^{-x})$
$= 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$
$= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$

21. $\because f(x_0)=f(x_1)=f(x_2)=\cdots=f(x_n)=0$

By Rolle's thm, $\exists\, x_{1i}$'s, $i=0,1,2,\cdots,n-1$
s.t. $x_{1i}\in(x_i,x_{i+1})$, $\forall i=0,1,2,\cdots,n-1$
$(x_0<x_{10}<x_1<x_{11}<x_2<x_{12}<x_3<x_{13}<\cdots)$
& $f'(x_{1i})=0$, $\forall i=0,1,2,\cdots,n-1$

By this & Rolle's thm, $\exists x_{2i}$'s, $i=0,1,\cdots,n-2$
s.t. $x_{2i}\in(x_{1i},x_{1i+1})$, $\forall i=0,1,2,\cdots,n-2$
& $f^{(2)}(x_{2i})=0$, $\forall i=0,1,2,\cdots,n-2$

$\vdots$

$\exists\, x_{n,0}$ s.t. $x_{n,0}\in(x_{n-1,0},x_{n-1,1})$
& $f^{(n)}(x_{n0})=0$

Set $\xi\equiv x_{n0}$. Then $\xi\in(x_0,x_n)$
& $f^{(n)}(\xi)=0$

23. $\sin x = x-\frac{\cos\xi}{6}x^3$
for some $\xi\in I(0,x)$

$\therefore |x-\sin x| = \left|\frac{\cos\xi}{6}x^3\right|$
$\leq \frac{1}{6}|x|^3$

correct to six decimal places
$\Leftrightarrow |\text{Error}| \leq \frac{1}{2}10^{-6}$
$\Leftrightarrow \frac{|x|^3}{6} \leq \frac{1}{2}\cdot 10^{-6}$
$\Leftrightarrow |x|^3 \leq 3\cdot 10^{-6}$
$\therefore |x| \leq \sqrt[3]{3}\cdot 10^{-2} \approx 1.44\times 10^{-2}$
$\therefore -0.0144 \leq x \leq 0.0144$
$(\text{or, } -\sqrt[3]{3}\cdot 10^{-2} \leq x \leq \sqrt[3]{3}\cdot 10^{-2})$

25. $e^x = 1+x+\frac{e^\xi}{2}x^2$
for some $\xi\in I(0,x)$
Since $\frac{e^\xi}{2}x^2 > 0$, $\forall x\neq 0$
$\therefore e^x > 1+x$, $\forall x\neq 0$

31. $f(x) = e^{\cos x}$
$f'(x) = -\sin x\, e^{\cos x}$
$f''(x) = -\cos x\, e^{\cos x} + \sin^2 x\, e^{\cos x}$
$\therefore f(\pi) = e^{-1}$, $f'(\pi)=0$
$f''(\pi) = e^{-1}$
$\therefore \underline{P_2(x) = e^{-1} + \frac{e^{-1}}{2}(x-\pi)^2}$