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Lab 7

1. $A(t) = \begin{pmatrix} t^2+1 & -\sqrt{2t+1} \\ \frac{1}{t-2} & -1 \end{pmatrix}$

$\vec f(t) = \begin{pmatrix} 3t \\ \ln(\cos t)\end{pmatrix}$

$A(t)$: conti if $t \geq -\frac{1}{2}$ & $t \neq 2$

$\vec f(t)$: conti if $\cos t > 0$

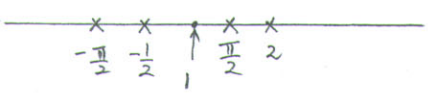
Note that

$\cos t > 0$

$\Leftrightarrow (-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, \frac{5\pi}{2}) \cup \dots$

Since $t_0 = 1$,

$I = (-\frac{1}{2}, \frac{\pi}{2})$



2. (a) $\vec x'(t) = A\vec x$, where

$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 + 4$

∴ ch. eq : $\lambda^2 - 2\lambda + 5 = 0$

∴ $\lambda = 1 \pm 2i$

Let $\lambda_1 = 1 - 2i$. Then

$(A - \lambda_1 I)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2ai - 2b = 0$

If we choose $a = 1$, then $b = i$

∴ $\lambda_1 = 1 - 2i$, $\vec V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\vec x_1(t) = \vec V_1 e^{\lambda_1 t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1-2i)t}$

$= \begin{pmatrix} 1 \\ i \end{pmatrix} e^t(\cos 2t - i\sin 2t)$

$= e^t \begin{pmatrix} \cos 2t - i\sin 2t \\ i\cos 2t + \sin 2t \end{pmatrix}$

∴ $\vec x_1(t) = \underbrace{e^t\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}} + i\underbrace{e^t\begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}}$

∴ $\vec x(t) = c_1 e^t\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} + c_2 e^t\begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}$

$= \begin{pmatrix} c_1 e^t\cos 2t - c_2 e^t \sin 2t \\ c_1 e^t \sin 2t + c_2 e^t \cos 2t \end{pmatrix}$

$\vec x(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = -3 \end{array}$

∴ $\vec x(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 3e^t \sin 2t \\ -3e^t\cos 2t \end{pmatrix}$

∴ $x(t) = 3e^t\sin 2t$, $y(t) = -3e^t\cos 2t$

(b) $\vec x' = A\vec x$, $A = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$

$\vec x(t) = e^{At}\vec x_0$, $\vec x_0 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$At = \begin{pmatrix} -t & 2t \\ 0 & -t \end{pmatrix} = \underbrace{\begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix}}_{\equiv B} + \underbrace{\begin{pmatrix} 0 & 2t \\ 0 & 0 \end{pmatrix}}_{\equiv C}$

Since $B = -tI$, $BC = CB$

∴ $e^{At} = e^{B+C} = e^B e^C$

$e^B = e^{\begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix}} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix}$

Since $C^n = \mathbb{O}$, $\forall n \geq 2$,

$e^C = I + C = \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}$

∴ $e^{At} = e^B e^C = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} e^{-t} & 2te^{-t} \\ 0 & e^{-t} \end{pmatrix}$

$\vec x(t) = e^{At}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3e^{-t} - 4te^{-t} \\ -2e^{-t} \end{pmatrix}$

∴ $x(t) = 3e^{-t} - 4te^{-t}$

$y(t) = -2e^{-t}$

(c) $\vec x' = A\vec x$, $A = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}$

$|A - \lambda I| = 0 \Rightarrow \lambda^2 + 3\lambda = 0$.

∴ $\lambda = 0, -3$

(i) $\lambda_1 = 0$ ...... $\vec V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(ii) $\lambda_2 = -3$ ...... $\vec V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

∴ $\vec x(t) = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}\overset{1}{\cancel{e^{0t}}} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t}$

$= \begin{pmatrix} 2c_1 + c_2 e^{-3t} \\ c_1 - c_2 e^{-3t} \end{pmatrix}$

$\lim_{t\to\infty} \frac{x(t)}{y(t)} = \lim_{t\to\infty} \frac{2c_1 + c_2 e^{-3t}}{c_1 - c_2 e^{-3t}}$

$= \frac{2c_1}{c_1} = 2$

3. (a) Since $x^2 + y^2 \geq 0$

$x' = x^2 + y^2 + 1 \geq 1$, $x(0) = 1$

∴ $\lim_{t\to\infty} x(t) = \infty$

$\Rightarrow y'(t) = x(t) \geq 1$ for all large t

∴ $\lim_{t\to\infty} y(t) = \infty$

(b) Let $z(t) \equiv x(t) + y(t)$

$\Rightarrow z'(t) = x' + y' = (2x + y + 3) + (-2x - y - 1) = 2$

$z(0) = x(0) + y(0) = 3$

∴ $z(t) = 2t + 3$

∴ $x(5) + y(5) = z(5) = 13$

4. $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$, $\vec x_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\vec x(t) = e^{At}\vec x_0$

$At = \begin{pmatrix} 2t & 0 \\ t & 2t \end{pmatrix} = \begin{pmatrix} 2t & 0 \\ 0 & 2t \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \equiv B + C$

Since $B = 2tI$, $BC = CB$,

$e^{At} = e^{B+C} = e^B e^C$

$e^B = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix}$

$e^C = I + C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$

∴ $e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ te^{2t} & e^{2t} \end{pmatrix}$

∴ $\vec x(t) = \begin{pmatrix} e^{2t} & 0 \\ te^{2t} & e^{2t} \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -e^{2t} \\ -te^{2t} + e^{2t} \end{pmatrix}$

∴ $\begin{cases} x(t) = -e^{2t} \\ y(t) = -te^{2t} + e^{2t} \end{cases}$