Lab 4

1. $\frac{dx}{dt} = r_i c_i - \frac{r_o}{V} x$ $\quad \begin{pmatrix} x(0) = 0 \\ V(0) = 100 \end{pmatrix}$

$c_i = 1$, $r_i = 2$, $r_o = 3$

$\therefore V(t) = 100 - t \quad (0 \le t \le 100)$

$\therefore \frac{dx}{dt} + \frac{3}{100-t} x = 2$

$\mu(t) = e^{3\int \frac{dt}{100-t}} = e^{-3\ln(100-t)} = (100-t)^{-3}$

$\therefore \frac{d}{dt}\left[x(100-t)^{-3}\right] = 2(100-t)^{-3}$

$\therefore x(100-t)^{-3} = (100-t)^{-2} + C$

$\therefore x = (100-t) + C(100-t)^3$

$x(0) = 0 \to 0 = 100 + C\,100^3$

$\therefore C = -10^{-4} = -\frac{1}{10000}$

$\therefore x(t) = (100-t) - \frac{(100-t)^3}{10^4} \quad (0 \le t \le 100)$

$x'(t) = -1 + \frac{3}{10^4}(100-t)^2 \equiv 0$

$\to (100-t)^2 = \frac{10^4}{3} \to 100 - t = \pm \frac{100}{\sqrt{3}}$

$\therefore t = 100 \pm \frac{100}{\sqrt{3}}$

Since $100 + \frac{100}{\sqrt{3}} > 100$ (out of domain)

Compare $x(0)$, $x(100 - \frac{100}{\sqrt{3}})$, $x(100)$

$x(0) = x(100) = 0$

$x(100 - \frac{100}{\sqrt{3}}) = \frac{100}{\sqrt{3}} - \frac{1}{10^4}\left(\frac{100}{\sqrt{3}}\right)^3$

$= \frac{100}{\sqrt{3}} - \frac{100}{3\sqrt{3}} = \frac{200}{3\sqrt{3}} = \frac{200\sqrt{3}}{9} > 0$

$\therefore x(t)$ attains its maximum at

$t = 100 - \frac{100}{\sqrt{3}} \approx 42.265$ (min)

2. $\frac{dP}{dt} = (\beta - \delta)P = (8 - 0.002P)P$

$= 0.002(4000 - P)M$

$\to$ logistic eq with $k = 0.002$, $M = 4000$

$\therefore \lim_{t\to\infty} P(t) = M = 4000$

(or use the formula

$P(t) = \frac{4000(1000)}{1000 + 3000e^{-8t}}$

to get $\lim_{t\to\infty} P(t) = \frac{4000(1000)}{1000} = 4000$)

3. $\begin{cases} \text{population unit} & : 1000 \text{ people} \\ \text{time unit} & : \text{day} \end{cases}$

$\frac{dN}{dt} = kN\cdot\underbrace{(15-N)}_{\text{\# of people who don't have the deasease}}$ —— $(*)$

$N(0) = 5$, $\frac{dN}{dt}(0) = 0.5$ —— $(**)$

$\therefore 0.5 = k\,5\cdot(15-5)$ $\quad \Leftarrow (*) \,\&\, (**)$

$\therefore 0.5 = k\,50 \quad \therefore k = \frac{1}{100}$

$\therefore (*)$ is a logistic equation with

$k = \frac{1}{100}$ & $M = 15$

$\therefore N(t) = \frac{75}{5 + 10e^{-0.15t}}$

Solve $N(t) = 10$ for t

$\cancel{10}\,2 = \frac{\cancel{75}\,15}{5 + 10\cdot e^{-0.15t}} \to 10 + 20\cdot e^{-0.15t} = 15$

$\therefore e^{-0.15t} = \frac{1}{4} \to -0.15t = -\ln 4$

$\therefore t = \frac{\ln 4}{0.15} \approx 9.24$ (days)

4. $\beta(t) = \frac{k}{\sqrt{P}}$, $\delta(t) = \frac{2}{1000}$

$\frac{dP}{dt} = (\beta(t) - \delta(t))P(t)$, $P(0) = 10^4$, $P'(0) = 20$

$P' = \left(\frac{k}{\sqrt{P}} - \frac{2}{1000}\right)P$

$\therefore 20 = \left(\frac{k}{\sqrt{10000}} - \frac{2}{1000}\right)10^4 = 100k - 20$

$\therefore k = \frac{40}{100} = \frac{2}{5} \to P' = \left(\frac{2}{5}\frac{1}{\sqrt{P}} - \frac{2}{10^3}\right)P$

$\therefore \frac{dP}{dt} + \frac{2}{10^3}P = \frac{2}{5}P^{\frac{1}{2}}$: Bernoulli eq with $n = \frac{1}{2}$

$\therefore v = P^{1-\frac{1}{2}} = P^{\frac{1}{2}} \quad \therefore \frac{dv}{dt} = \frac{1}{2}P^{-\frac{1}{2}}\frac{dP}{dt}$

$P^{-\frac{1}{2}}\frac{dP}{dt} + \frac{2}{10^3}P^{\frac{1}{2}} = \frac{2}{5}$

$\therefore 2\frac{dv}{dt} + \frac{2}{10^3}v = \frac{2}{5} \to \frac{dv}{dt} + \frac{1}{10^3}v = \frac{1}{5}$

$\therefore \mu(t) = e^{\int \frac{1}{10^3}dt} = e^{\frac{t}{10^3}}$

$\therefore \frac{d}{dt}\left[e^{\frac{t}{1000}}v\right] = \frac{1}{5}e^{\frac{t}{1000}}$

$\therefore v e^{\frac{t}{1000}} = 200e^{\frac{t}{1000}} + C$

$\therefore P^{\frac{1}{2}} = v = 200 + Ce^{-\frac{t}{1000}}$

$\therefore P(t) = \left(200 + Ce^{-\frac{t}{1000}}\right)^2$

$P(0) = 10^4 \to 10^4 = (200 + C)^2 \to C = -100$

$\therefore P(t) = \left(200 - 100e^{-\frac{t}{1000}}\right)^2$

5. $\beta(t) = \frac{k_1}{\sqrt{P}}$, $\delta(t) = \frac{k_2}{\sqrt{P}}$

(a) $\frac{dP}{dt} = (\beta(t) - \delta(t))P = \frac{k}{\sqrt{P}}P$

$= k\sqrt{P} \quad (k = k_1 - k_2)$

$\int \frac{dP}{\sqrt{P}} = \int k\,dt \to 2\sqrt{P} = kt + C$

$P(0) = P_0 \to 2\sqrt{P_0} = C$

$\therefore \sqrt{P} = \frac{1}{2}kt + \frac{C}{2} = \frac{1}{2}kt + \sqrt{P_0}$

$\therefore P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\right)^2$

(b) $P_0 = 100 \to P(t) = \left(\frac{kt}{2} + 10\right)^2$

$P(6) = (3k + 10)^2 = 169 = 13^2$

$\therefore 3k + 10 = 13 \to k = 1$ (the time unit is a month)

$\therefore P(t) = \left(\frac{t}{2} + 10\right)^2$

$\therefore P(12) = (6 + 10)^2 = 256$

6. $\frac{dP}{dt} = kP^2 \quad (t = 0 \leftrightarrow \text{yr } 1988)$

$P(0) = 12$, $P(10) = 24$

$\int \frac{dP}{P^2} = \int k\,dt \to -\frac{1}{P} = kt + C$

$\therefore P(t) = \frac{1}{C - kt}$

$P(0) = 12 \to 12 = \frac{1}{C} \to C = \frac{1}{12}$

$\therefore P(t) = \frac{1}{\frac{1}{12} - kt} = \frac{12}{1 - 12kt}$

$P(10) = 24 \to \frac{\cancel{12}}{1 - 120k} = \cancel{24}\,2 \quad 1 - 120k = \frac{1}{2}$

$\therefore k = \frac{1}{240}$

$\therefore P(t) = \frac{12}{1 - \frac{12}{240}t} = \frac{12}{1 - \frac{t}{20}} = \frac{240}{20 - t}$

(a) $P(t) = \frac{240}{20-t} \equiv 48$

$\Rightarrow 20 - t = \frac{240}{48} = 5 \quad \therefore t = 15$

i.e., year 2003

(b) $\lim_{t\to 20^-} P(t) = \lim_{t\to 20^-} \frac{240}{20-t} = \infty$

i.e., the population will explode in 2008 !!