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Lab 3

1. (a) $\underbrace{(y\cos x + 2x\tan y)}_{M}dx + \underbrace{(\sin x + x^2\sec^2 y)}_{N}dy = 0$

$$\frac{\partial M}{\partial y} = \cos x + 2x\sec^2 y = \frac{\partial N}{\partial x}$$

$$\frac{\partial F}{\partial x} = y\cos x + 2x\tan y$$

$$\to F(x,y) = y\sin x + x^2\tan y + g(y)$$

$$\frac{\partial F}{\partial y} = \sin x + x^2\sec^2 y + g'(y)$$

$$= \sin x + x^2 \sec^2 y \quad \Rightarrow g'(y) = 0$$

$$\therefore g(y) = c$$

$\therefore$ Solution: $y\sin x + x^2\tan y = c$

(b) $\underbrace{y}_{M}dx + \underbrace{(2x^2y^2 - x)}_{N}dy = 0$

$\to$ Not exact!

$$\frac{M_y - N_x}{N} = \frac{1-4xy^2+1}{2x^2y^2 - x} = \frac{-2(2xy^2-1)}{x(2xy^2-1)} = -\frac{2}{x}$$

$$\therefore \frac{d\mu}{dx} = -\frac{2}{x}\mu \to \int\frac{d\mu}{\mu} = \int -\frac{2}{x}dx$$

$$\therefore \ln\mu = -2\ln x \to \mu(x) = \frac{1}{x^2}$$

$\therefore \frac{y}{x^2}dx + (2y^2 - \frac{1}{x})dy = 0 \to$ exact

$$\frac{\partial F}{\partial x} = \frac{y}{x^2} \to F(x,y) = -\frac{y}{x} + g(y)$$

$$\frac{\partial F}{\partial y} = -\frac{1}{x} + g'(y) = 2y^2 - \frac{1}{x} \to g'(y) = 2y^2$$

$$\therefore g(y) = \frac{2}{3}y^3$$

$\therefore$ Solution: $-\frac{y}{x} + \frac{2}{3}y^3 = c$

(c) $xy' = y + xe^{\frac{y}{x}}$

$$\frac{dy}{dx} = \frac{y}{x} + e^{\frac{y}{x}}, \quad \frac{y}{x} = v \to y = xv$$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \cancel{v} + x\frac{dv}{dx} = \cancel{v} + e^v$$

$$\int e^{-v}dv = \int\frac{dx}{x} \to -e^{-v} = \ln|x| + c$$

$$\therefore e^{-v} = c - \ln|x|$$

$$-\frac{y}{x} = -v = \ln(c - \ln|x|)$$

$$\therefore y = -x\ln(c - \ln|x|)$$

2. (a) $\frac{dy}{dx} + \frac{3}{x}y = 3y^{\frac{4}{3}}$ — (*)

$\to$ Bernoulli eq with $n = \frac{4}{3}$

$$\therefore v \equiv y^{1-\frac{4}{3}} = y^{-\frac{1}{3}}$$

$$\frac{dv}{dx} = -\frac{1}{3}y^{-\frac{4}{3}}\frac{dy}{dx}$$

$$(*) \to \quad y^{-\frac{4}{3}}\frac{dy}{dx} + \frac{3}{x}y^{-\frac{1}{3}} = 3$$

$$\therefore -3\frac{dv}{dx} + \frac{3}{x}v = 3$$

$$\therefore \frac{dv}{dx} - \frac{1}{x}v = -1$$

$$\mu(x) = e^{-\int\frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$$

$$\therefore \frac{d}{dx}\left[\frac{1}{x}v\right] = -\frac{1}{x} \to \frac{1}{x}v = -\ln|x| + c$$

$$\therefore v = -x\cdot(\ln(-x) + c) \quad \left(\because x<0 \text{ around } x=-e\right)$$

$$\therefore y^{-\frac{1}{3}} = -x(\ln(-x) + c)$$

$$y = \left[-x\{\ln(-x) + c\}\right]^{-3}$$

$$\frac{1}{8e^3} = \left[e\{\ln e + c\}\right]^{-3} \quad y(-e) = \frac{1}{8e^3}$$

$$\therefore (2e)^{-3} = \left[e(1+c)\right]^{-3}$$

$$\therefore 2 = 1 + c \Rightarrow c = 1$$

$$\therefore y = \left[-x\{\ln(-x) + 1\}\right]^{-3}$$

(b) $y' = \frac{y}{x+y} = \frac{y/x}{1+y/x} \equiv \frac{v}{1+v}$

$$v = \frac{y}{x} \quad \therefore y = vx \to \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$v + x\frac{dv}{dx} = \frac{v}{v+1} \quad \therefore x\frac{dv}{dx} = \frac{\cancel{v} - v^2 - \cancel{v}}{v+1}$$

$$\therefore -\frac{v+1}{v^2}dv = \frac{dx}{x} \quad \therefore \left(-\frac{1}{v^2} - \frac{1}{v}\right)dv = \frac{dx}{x}$$

$$\frac{1}{v} - \ln v = \ln x + c \quad (\because x, y > 0 \to v > 0)$$

$$\frac{1}{v} = \ln v + \ln x + c = \ln(vx) + c$$

$$\frac{x}{y} = \ln y + c \quad \xrightarrow{y(2)=1} \quad \frac{2}{1} = \cancel{\ln 1}^{\,0} + c$$

$$\therefore c = 2$$

$$\therefore \frac{x}{y} = \ln y + 2$$

$$\therefore x = y(\ln y + 2)$$

(c) $v \equiv x + y \to y = v - x \to \frac{dy}{dx} = \frac{dv}{dx} - 1$

$$y' = \sqrt{x+y} \to v' - 1 = \sqrt{v} \to \frac{dv}{dx} = \sqrt{v} - 1$$

$$\therefore \frac{dv}{\sqrt{v}-1} = dx \qquad v \equiv u^2 \to dv = 2u\,du$$

$$\frac{2u\,du}{u-1} = dx \to \frac{2(u+1)-2}{u+1}du = dx$$

$$\left(2 - \frac{2}{u+1}\right)du = dx \to 2u - 2\ln(u+1) = x + c$$

$$\therefore 2\sqrt{v} - 2\ln(\sqrt{v}+1) = x + c$$

$$\therefore 2\sqrt{x+y} - 2\ln(\sqrt{x+y}+1) = x + c$$

$$y(1) = 0 \to 2 - 2\ln 2 = 1 + c$$

$$\therefore c = 1 - 2\ln 2$$

$$\therefore 2\sqrt{x+y} - 2\ln(\sqrt{x+y}+1) = x + 1 - 2\ln 2$$

3. $\frac{dy}{dx} = \frac{xy - 2y}{x^3}, \quad y(1) = e$

$$\therefore x^3\frac{dy}{dx} = y(x-2) \to \frac{dy}{y} = \left(\frac{1}{x^2} - \frac{2}{x^3}\right)dx$$

$$\ln|y| = -\frac{1}{x} + \frac{1}{x^2} + c, \quad y(1) = e$$

$$\ln e = -1 + 1 + c \qquad \therefore c = 1$$

$$\therefore \ln y = -\frac{1}{x} + \frac{1}{x^2} + 1$$

$$\therefore y = e^{-\frac{1}{x} + \frac{1}{x^2} + 1}$$

4. (a) Eq (5), on p69:

$$(5): \frac{\partial\mu}{\partial y}M - \frac{\partial\mu}{\partial x}N = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu$$

Suppose $\mu(x,y) = \mu(z)$ where $z = xy$

Then, $\frac{\partial\mu}{\partial y} = \frac{d\mu}{dz}x$, $\frac{\partial\mu}{\partial x} = \frac{d\mu}{dz}y$

$$\therefore (5) \Leftrightarrow \frac{d\mu}{dz}(xM - yN) = (N_x - M_y)\mu \quad \text{— (*)}$$

$$\therefore \frac{N_x - M_y}{xM - yN} = \frac{\frac{d\mu}{dz}}{\mu} = H(z)$$

H is a ftn

$$\therefore \frac{N_x - M_y}{xM - yN} = H(xy) \quad \text{— (**)}$$

Conversely, if (**) holds, let $\rho(z) \equiv e^{\int H(z)dz}$ and define $\mu(x,y) = \rho(xy)$.

It is easy to show that this $\mu(x,y)$ satisfy (5).

(b) $(xy^2e^{xy} - x)dx + (x^2ye^{xy} + \frac{x^2}{y})dy = 0$ — (***)

$$N_x = 2xye^{xy} + x^2y^2e^{xy} + \frac{2x}{y}$$

$$M_y = 2xye^{xy} + x^2y^2e^{xy}$$

$$xM - yN = x^2y^2e^{xy} - x^2 - x^2y^2e^{xy} - x^2 = -2x^2$$

$$\therefore \frac{N_x - M_y}{xM - yN} = \frac{\frac{2x}{y}}{-2x^2} = -\frac{1}{xy} \equiv H(xy)$$

$$\therefore H(z) = -\frac{1}{z} \to \rho(z) = e^{-\int\frac{dz}{z}} = e^{-\ln z} = \frac{1}{z}$$

$$\therefore \mu(x,y) = \frac{1}{xy}$$

$$(5) \to (ye^{xy} - \frac{1}{y})dx + (xe^{xy} + \frac{x}{y^2})dy = 0$$

$$\frac{\partial F}{\partial x} = ye^{xy} - \frac{1}{y} \to F(x,y) = e^{xy} - \frac{x}{y} + g(y)$$

$$\therefore \frac{\partial F}{\partial y} = xe^{xy} + \frac{x}{y^2} + g'(y) \equiv xe^{xy} + \frac{x}{y^2}$$

$$\therefore g'(y) = 0 \qquad \therefore g(y) = c$$

Choose $F(x,y) = e^{xy} - \frac{x}{y}$

$\therefore$ Solution: $e^{xy} - \frac{x}{y} = c$