205

Lab 2

W 08

1. (a) $\int \frac{dx}{(x-2)(x-3)} = \int dt$

$\int \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx = t + C_1$

$\ln|x-3| - \ln|x-2| = t + C_1$

$\therefore \ln\left|\frac{x-3}{x-2}\right| = t + C_1$

$\left|\frac{x-3}{x-2}\right| = C_2 e^t \quad (C_2 = e^{C_1} > 0)$

$\therefore \frac{x-3}{x-2} = C_3 e^t \quad (C_3 = \pm C_2 \neq 0)$

$x(0) = 4 \to \frac{1}{2} = C_3$

$\therefore \frac{x-3}{x-2} = \frac{1}{2}e^t \to 2x - 6 = e^t x - 2e^t$

$\therefore (2 - e^t)x = 6 - 2e^t$

$\therefore \underline{x = \frac{6 - 2e^t}{2 - e^t}}$

(b) $\int \frac{y}{1+y^2} dy = \int \frac{\cos x}{1 + \sin^2 x} dx \quad u = \sin x$

$\therefore \frac{1}{2}\ln(1+y^2) = \tan^{-1}(\sin x) + C_1$

$\ln(1+y^2) = 2\tan^{-1}(\sin x) + C_2$

$1 + y^2 = C_3 e^{2\tan^{-1}(\sin x)}$

$y(\frac{\pi}{2}) = 3 \to 10 = C_3 e^{2\tan^{-1}(1)} = C_3 e^{\frac{\pi}{2}}$

$\therefore C_3 = 10 e^{-\frac{\pi}{2}}$

$\therefore y = \pm\sqrt{10 e^{-\frac{\pi}{2}} e^{2\tan^{-1}(\sin x)} - 1}$

Since $y_0 = 3 > 0$, we choose

$\underline{y = \sqrt{10 \cdot e^{2\tan^{-1}(\sin x) - \frac{\pi}{2}} - 1}}$

(c) $\frac{dy}{dx} - (x^2+1)y = x^2 + 1$

$\mu(x) = e^{-\int (x^2+1)dx} = e^{-\frac{1}{3}x^3 - x}$

$\therefore \frac{d}{dx}\left[e^{-\frac{1}{3}x^3 - x} \cdot y\right] = (x^2+1)e^{-\frac{1}{3}x^3 - x}$

$\therefore e^{-\frac{1}{3}x^3 - x} \cdot y = \int (x^2+1) e^{-\frac{1}{3}x^3 - x} dx$

$= -e^{-\frac{1}{3}x^3 - x} + C$

$\therefore y = -1 + Ce^{\frac{1}{3}x^3 + x}$

$y(0) = 1 \Rightarrow 1 = -1 + C \quad \therefore C = 2$

$\therefore \underline{y = 2e^{\frac{1}{3}x^3 + x} - 1}$

(d) $(3x^2 + y^2)dx + (2xy + 3y^2)dy = 0$

$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x} \to$ exact eq

$\frac{\partial F}{\partial x} = 3x^2 + y^2 \to F(x,y) = x^3 + xy^2 + g(y)$

$\frac{\partial F}{\partial y} = 2yx + g'(y) \equiv 2xy + 3y^2 \to g'(y) = 3y^2$

$\therefore g(y) = y^3 + C \to F(x,y) = x^3 + xy^2 + y^3$

$y(2) = 1 \to x^3 + xy^2 + y^3 = C \to C = 11$

$\therefore \underline{x^3 + xy^2 + y^3 = 11}$

2. $\frac{dT}{dt} = k(M - T)$

(a) $\int \frac{dT}{M-T} = \int k\,dt \to -\ln|M-T| = kt + C_1$

$\ln|M-T| = -kt - C_1 \to |M-T| = C_2 e^{-kt}$

$\therefore M - T = C_3 e^{-kt} \quad (C_3 \neq 0)$

$\therefore T = M + Ce^{-kt}, \quad C \neq 0$

But $T \equiv M$ is a solution, too, which is covered by taking $C = 0$.

$\therefore \underline{T = M + Ce^{-kt}}$, $\underline{C \text{ is any real \#}}$

(b) $M = 70 \qquad \therefore T(t) = 70 + Ce^{-kt}$

$T(0) = 100 \to 100 = 70 + C \Rightarrow C = 30$

$\therefore T(t) = 70 + 30e^{-kt}$

$T(6) = 80 \to 80 = 70 + 30e^{-k6}$

$\therefore e^{-6k} = \frac{1}{3} \to -6k = \ln(\frac{1}{3}) = -\ln 3$

$\therefore k = \frac{\ln 3}{6}$

$\therefore T(t) = 70 + 30e^{-\frac{\ln 3}{6}t}$

$T(20) = 70 + 30e^{-\frac{\ln 3}{6} \cdot 20} \approx \underline{70.8}$

3. $\frac{dy}{dx} + \underbrace{\frac{x-3}{\ln(x-1)}}_{P(x)} y = \underbrace{\frac{x-3}{(2-3x)\ln(x-1)}}_{Q(x)}$

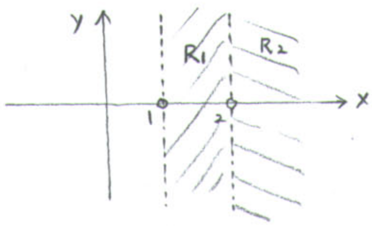
(a)

$\ln(x-1)$ is well-defined iff $x > 1$

$\ln(x-1) \neq 0$ iff $x \neq 2$

$2 - 3x \neq 0$ iff $x \neq \frac{2}{3}$

$\therefore P(x)$ & $Q(x)$ are continuous on $(1,2) \cup (2,\infty)$



$R_1 = (1,2) \times \mathbb{R}, \quad R_2 = (2,\infty) \times \mathbb{R}$

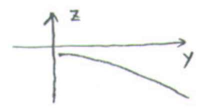
(b) (i) No solution, (ii) $(1,2)$

(iii) $(2,\infty)$, (iv) $(2,\infty)$

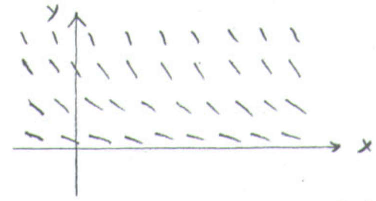
4. The slope of the tangent line to the solution curve $y = \eta(x)$ at a point (x,y) on the curve, (which is $f(y)$) is negative and it is decreasing (i.e., more negative) as y increases along the solution curve.

$\therefore z = f(y)$ is negative & decreasing ftn

$\therefore$



$\therefore$ The direction field is



(b) $\phi(x) = \eta(x+3)$: horizontal translation by -3

Then $\frac{d\phi}{dx} = \eta'(x+3)$ — ①

Also $\eta'(x) = f(\eta(x)), \ \forall x$

$\left(\because \eta \text{ is a solution to } \frac{dy}{dx} = f(y)\right)$

$\therefore \eta'(x+3) = f(\eta(x+3)); \ \forall x$ — ②

① & ② $\to \frac{d\phi}{dx} = f(\eta(x+3)) = f(\phi(x))$

$\therefore \phi(x)$ is a solution to $y' = f(y)$

(c) Note that $\phi(0) = \eta(3) = 5$

$\therefore$ Indeed, the ftn $\phi(x)$ defined in (b) is the solution to the IVP: $y' = f(y)$, $y(0) = 5$

$\therefore \phi(4) = \eta(7) = 2 \qquad (\because \phi(x) = \eta(x+3))$

$\phi(8) = \eta(11) = 1$

(d) Let ξ solve $y' = g(y)$, i.e.,

$\xi'(x) = g(\xi(x)), \ \forall x$

$\therefore \xi'(x-\alpha) = g(\xi(x-\alpha)), \ \forall x$

$\therefore \phi_\alpha'(x) = g(\phi_\alpha(x)), \ \forall x$

$\left(\begin{array}{l} \because \phi_\alpha(x) = \xi(x-\alpha) \\ \therefore \phi_\alpha'(x) = \xi'(x-\alpha) \end{array}\right)$

Thus, ϕ_α also a solution to $\frac{dy}{dx} = g(y)$

Graphical interpretation:

If $y = \xi(x)$ is a solution curve to the autonomous DE: $y' = g(y)$ then *any* horizontal translation of $y = \xi(x)$ provides another solution curve to the same DE.