2. $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

$0 = |A - rI| = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (1-r)(3-r)+1$

$= r^2 - 4r + 3 + 1 = (r-2)^2$

$\therefore r = 2$

$\therefore (A-2I)^2 = 0$ — $(*)$ $\quad \therefore k = 2$

$e^{At} = e^{[(A-2I)+2I]t} = e^{2It}e^{(A-2I)t} = e^{2t} I e^{(A-2I)t}$

$= e^{2t} e^{(A-2I)t}$

$= e^{2t}\left[I + (A-2I)t \right] \quad (\because \text{ by } (*))$

$= e^{2t}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & -t \\ t & 3t \end{bmatrix} + \begin{bmatrix} -2t & 0 \\ 0 & -2t \end{bmatrix} \right\}$

$= e^{2t}\begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{bmatrix}$

ch. eg : $(r+1)^3 = 0$

$\therefore r = -1$ and

$(A+I)^3 = 0 \quad \therefore k = 3$

$e^{At} = e^{-t} e^{(A+I)t}$

$= e^{-t}\left[I + (A+I)t + \frac{1}{2}(A+I)^2 t^2 \right]$

$= e^{-t}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 3 & -3 \end{bmatrix} + \frac{t^2}{2}\underbrace{\begin{bmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 3 & -3 \end{bmatrix}^2}_{= \begin{bmatrix} -3 & 0 & 1 \\ 0 & 0 & 0 \\ -9 & 0 & 3 \end{bmatrix}} \right\}$

$= e^{-t}\begin{bmatrix} 1+3t-\frac{3}{2}t^2 & t & -t+\frac{t^2}{2} \\ -3t & 1 & t \\ 9t-\frac{9}{2}t^2 & 3t & 1-3t+\frac{3}{2}t^2 \end{bmatrix}$